On Interval Global Dimension of Posets: a Characterization of Case 0

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神戸大学　人間発達環境学研究科

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

Joint work with

青木 利隆 氏(神戸)　エスカラ エマソン ガウ 氏(神戸)

Preprint Summand-injectivity of interval approximations and monotonicity of interval global dimension. Toshitaka Aoki, Emerson G. Escolar, Shunsuke Tada. arXiv:2308.14979.

発表の流れ

- ■位相的データ解析とは？
- ■定理(1)
- ■定理(2)

位相的データ解析(TDA)とは？

Topological Data Analysis

トポロジーを用いたデータ解析手法

位相的データ解析(TDA)とは？

Topological Data Analysis

トポロジーを用いたデータ解析手法

- パーシステントホモロジー解析
- Mapper解析
- topological flow analysis

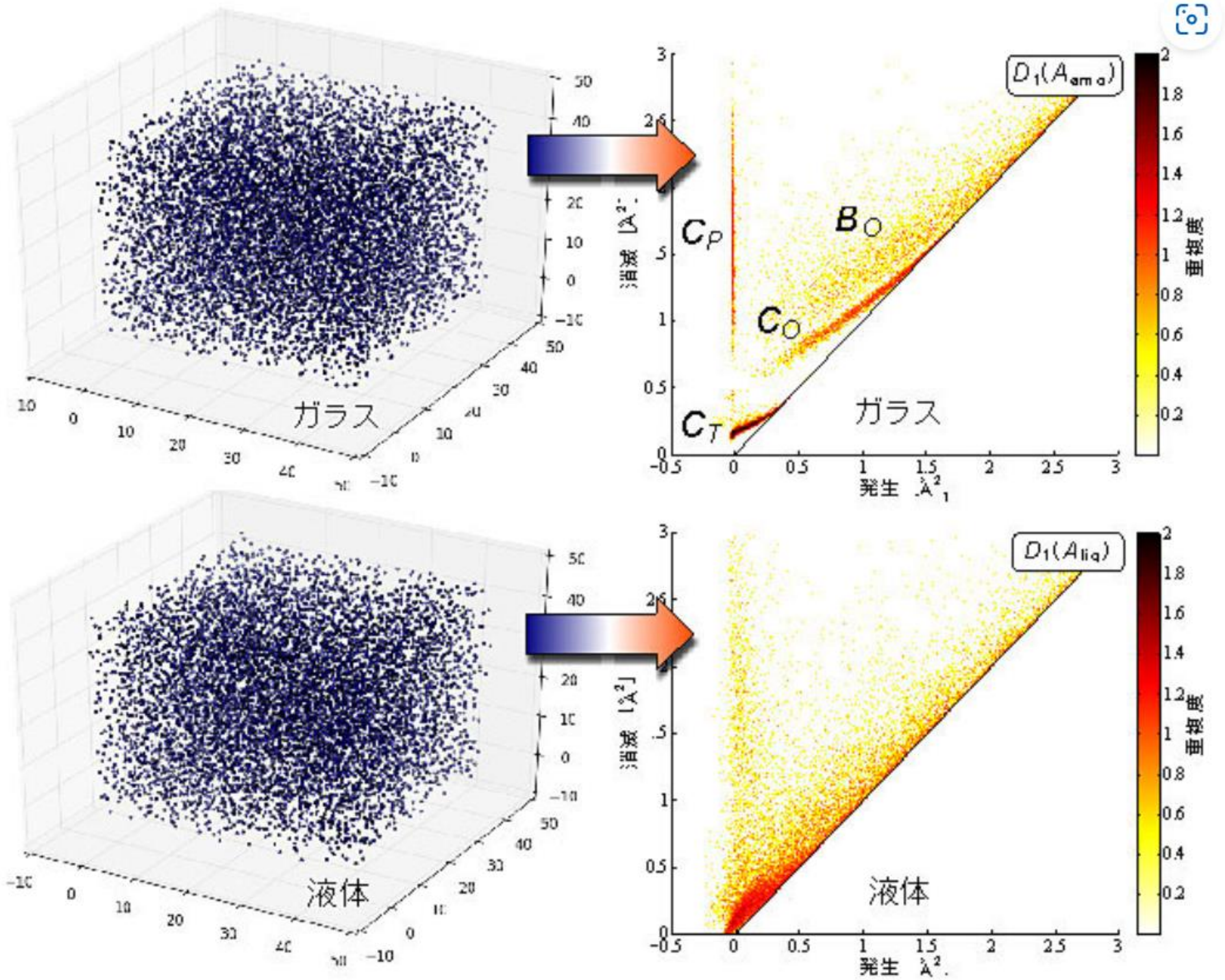

図１　ＳｉＯ２の原子配置（左）とそのパーシステントホモロジー（右）

[共同発表：ガラスの「形」を数学的に解明～トポロジーで読み解く無秩序の中の秩序～ (jst.go.jp)]

Hiraoka, Yasuaki, et al. "Hierarchical structures of amorphous solids characterized by persistent homology." *Proceedings of the National Academy of Sciences* 113.26 (2016): 7035-7040.

パーシステントホモロジー解析

データの形(穴や空洞)の
「パーシステンス」(持続性)
に着目

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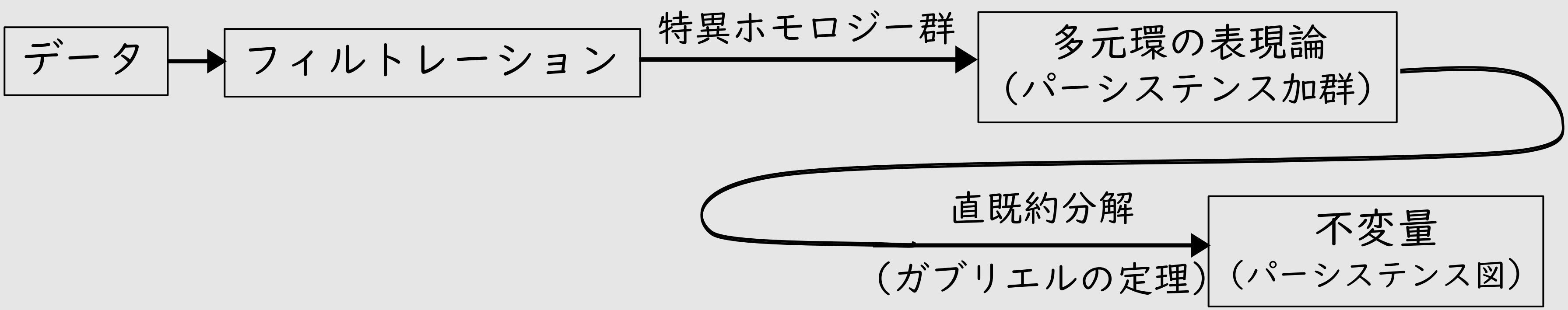

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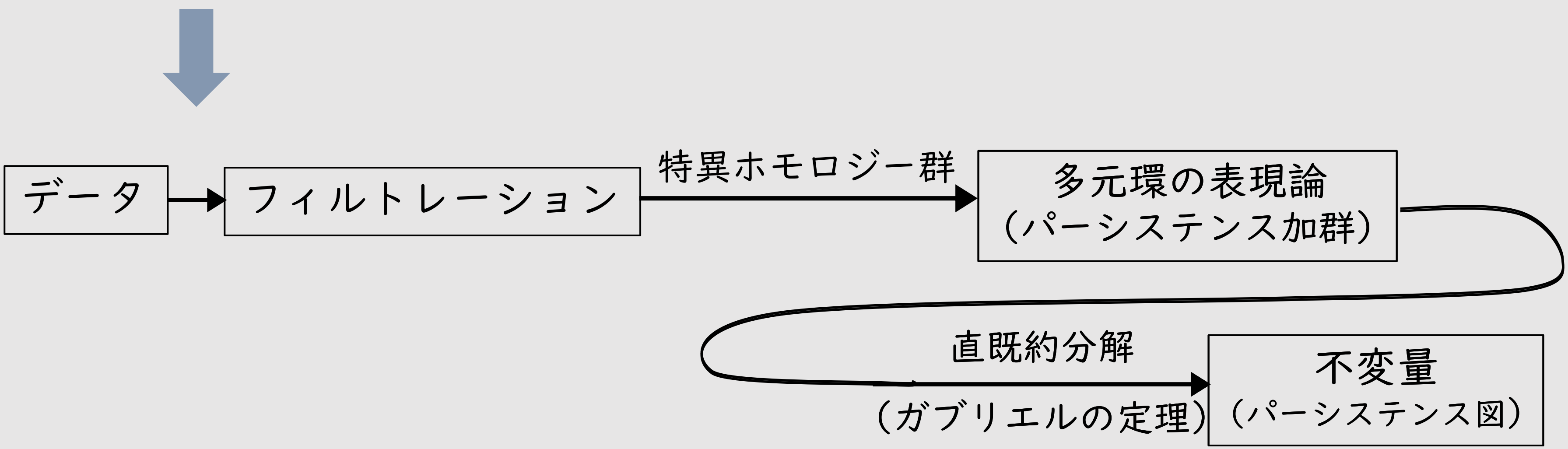
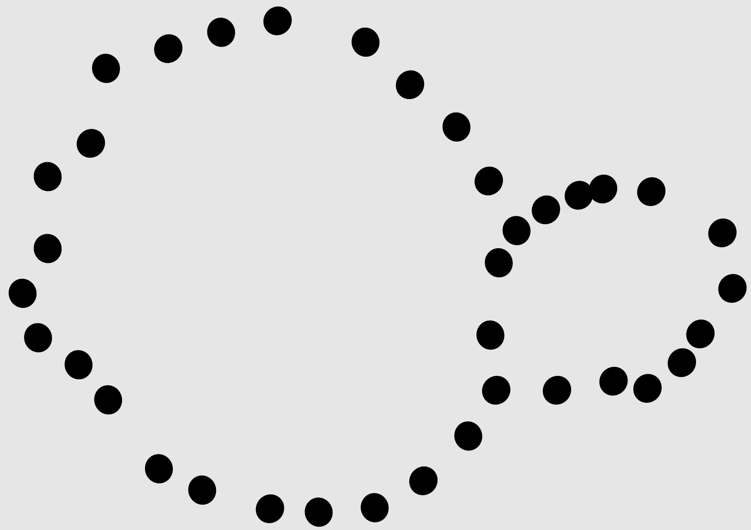
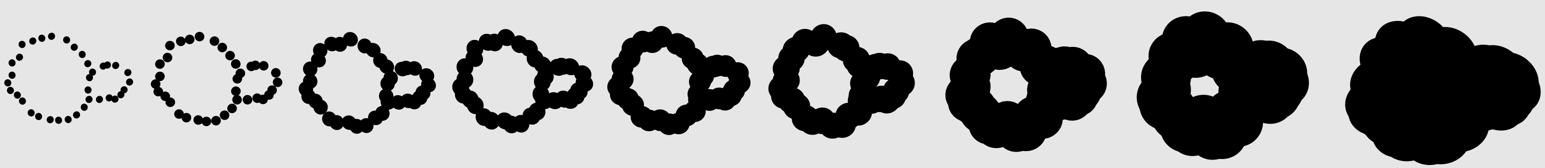


点群データから半径パラメータtを大きくすることにより

フィルトレーションを構成



t_0



t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8

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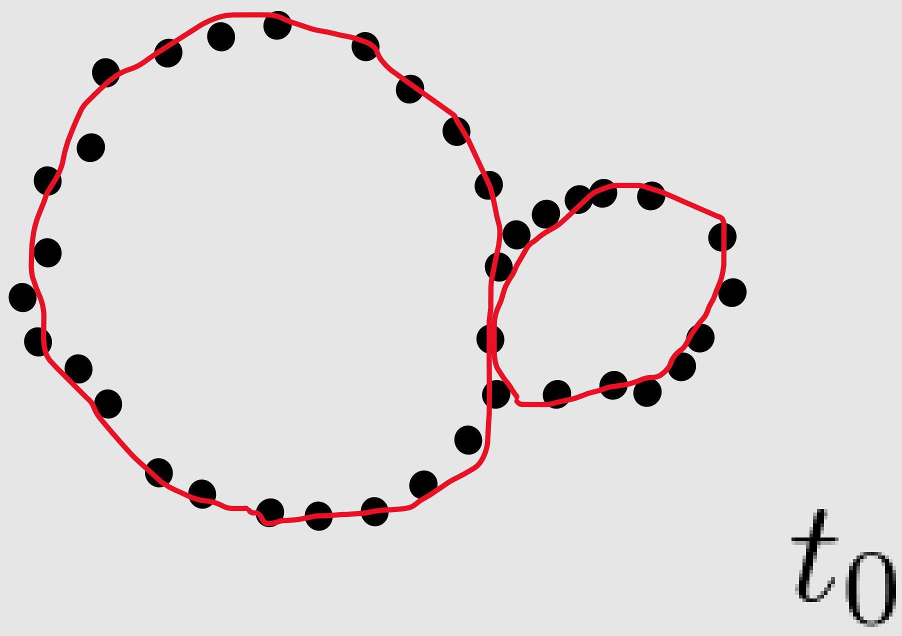

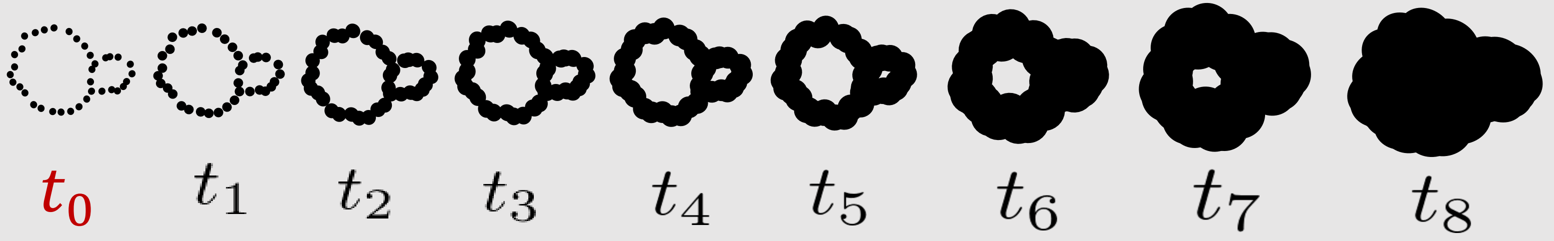

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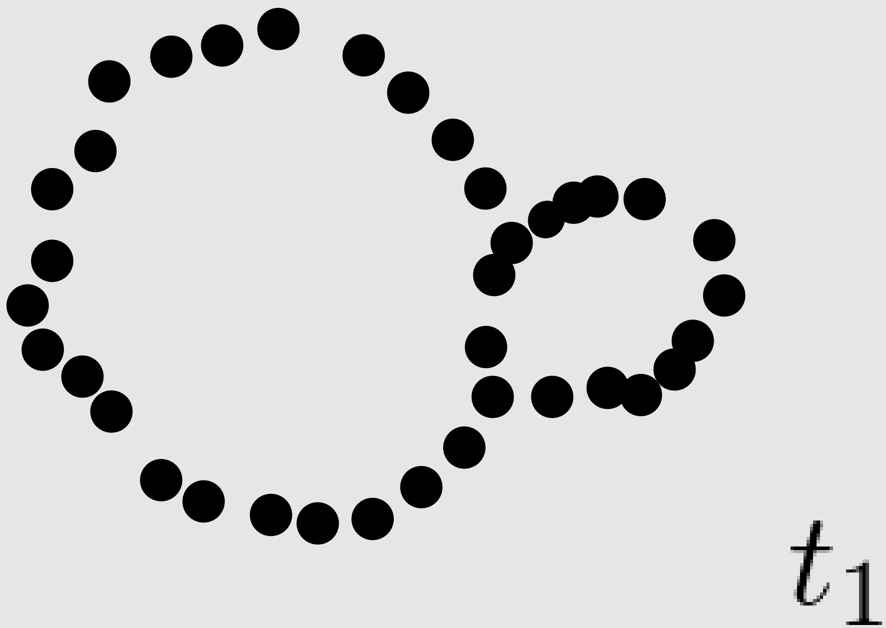

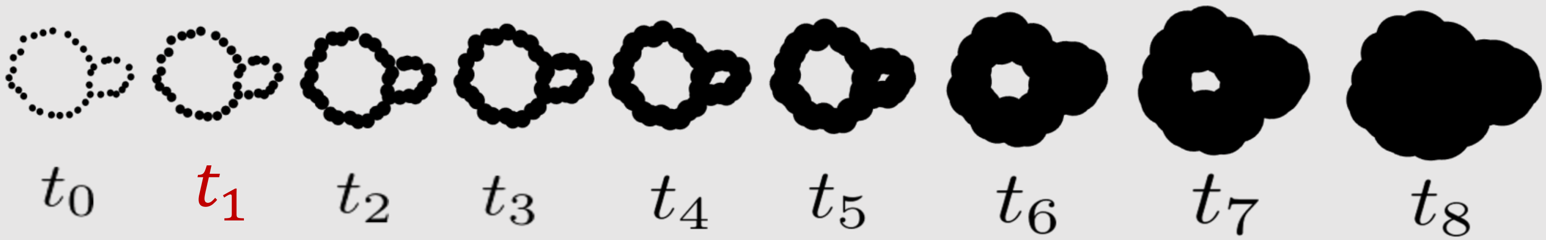

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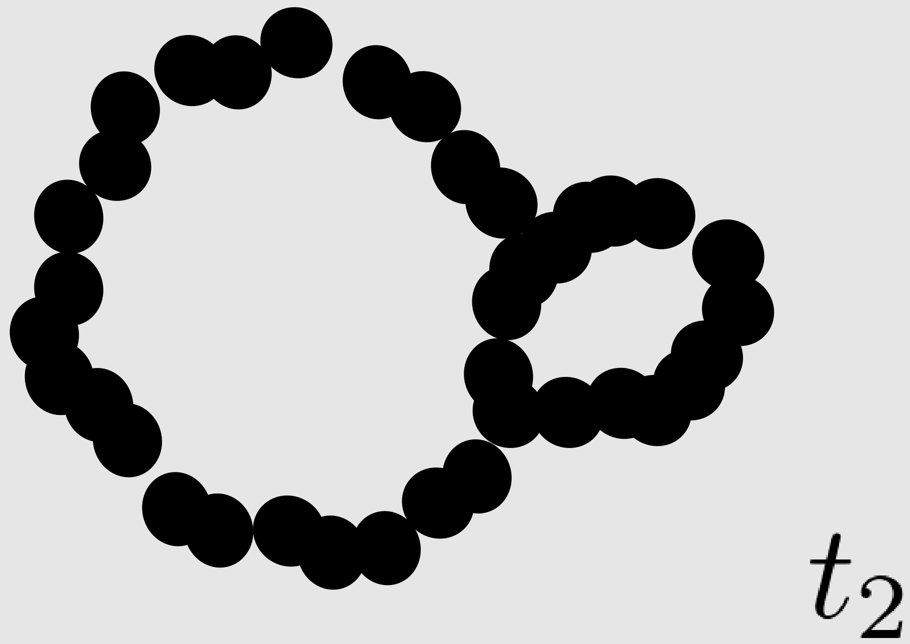

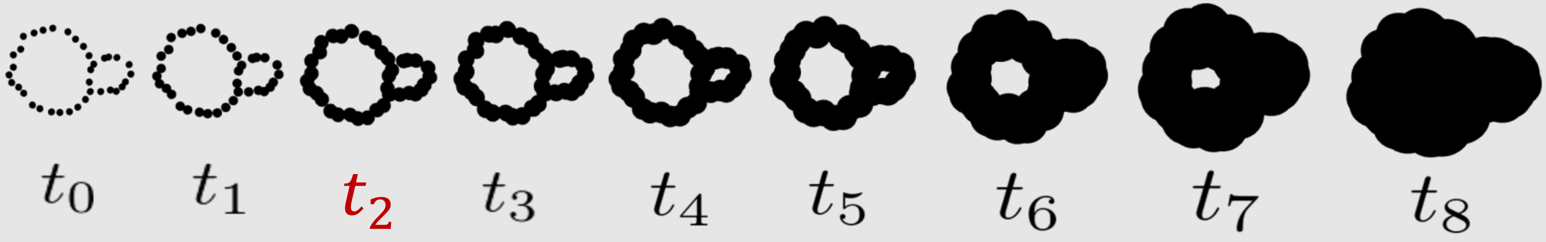

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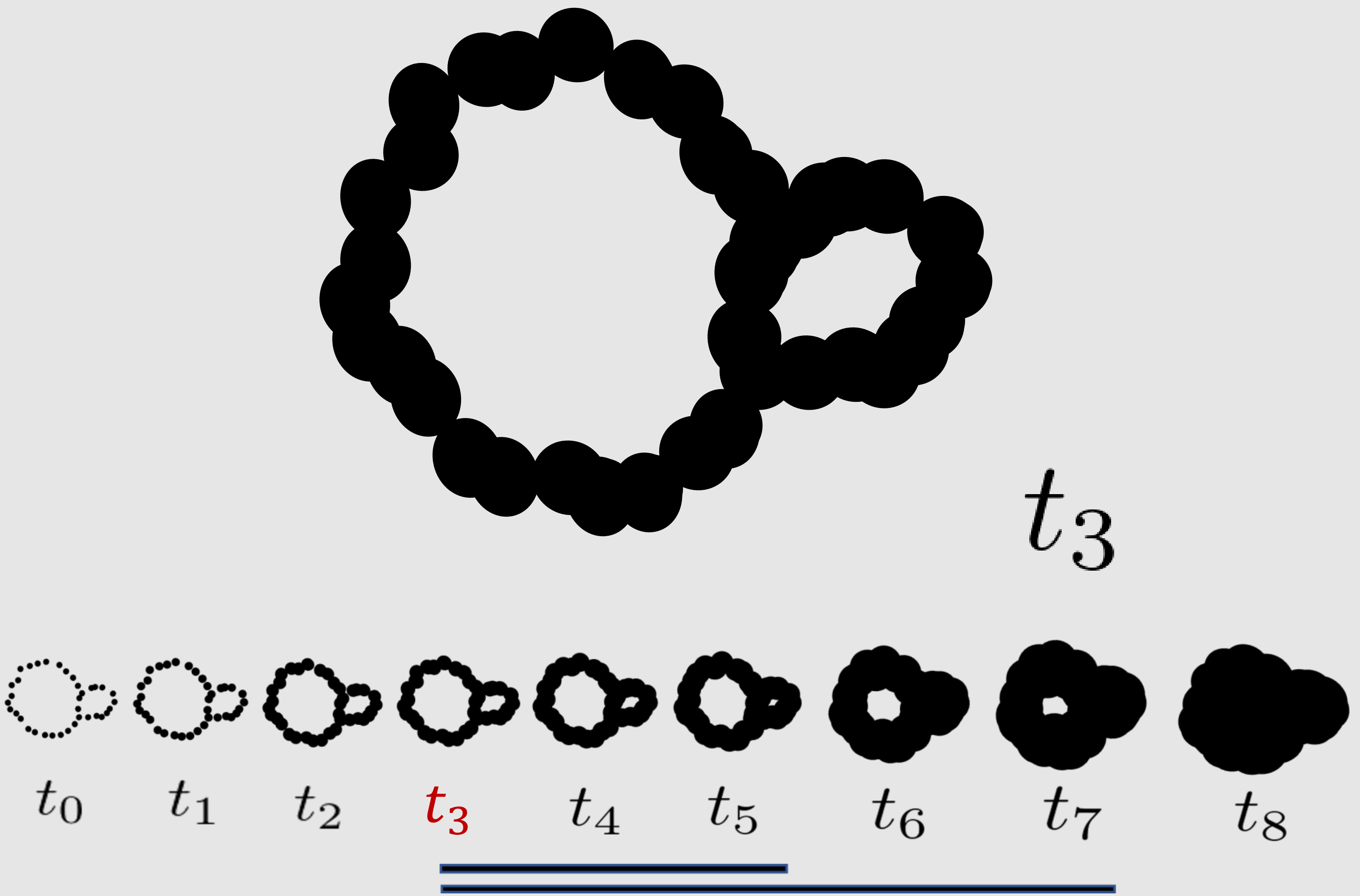

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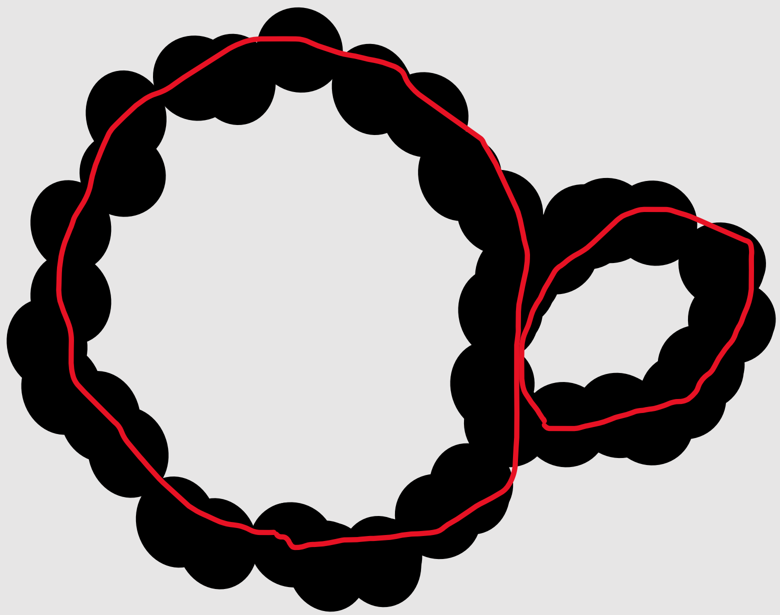
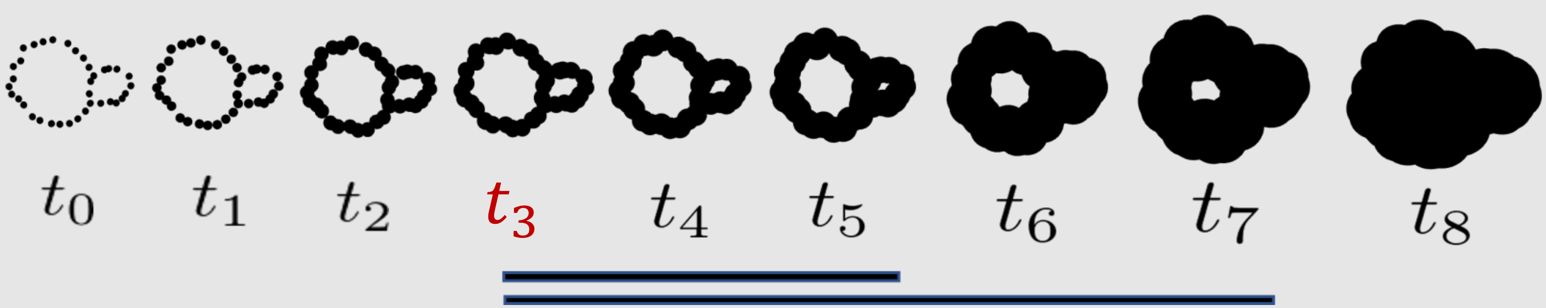


t_3　穴の生成



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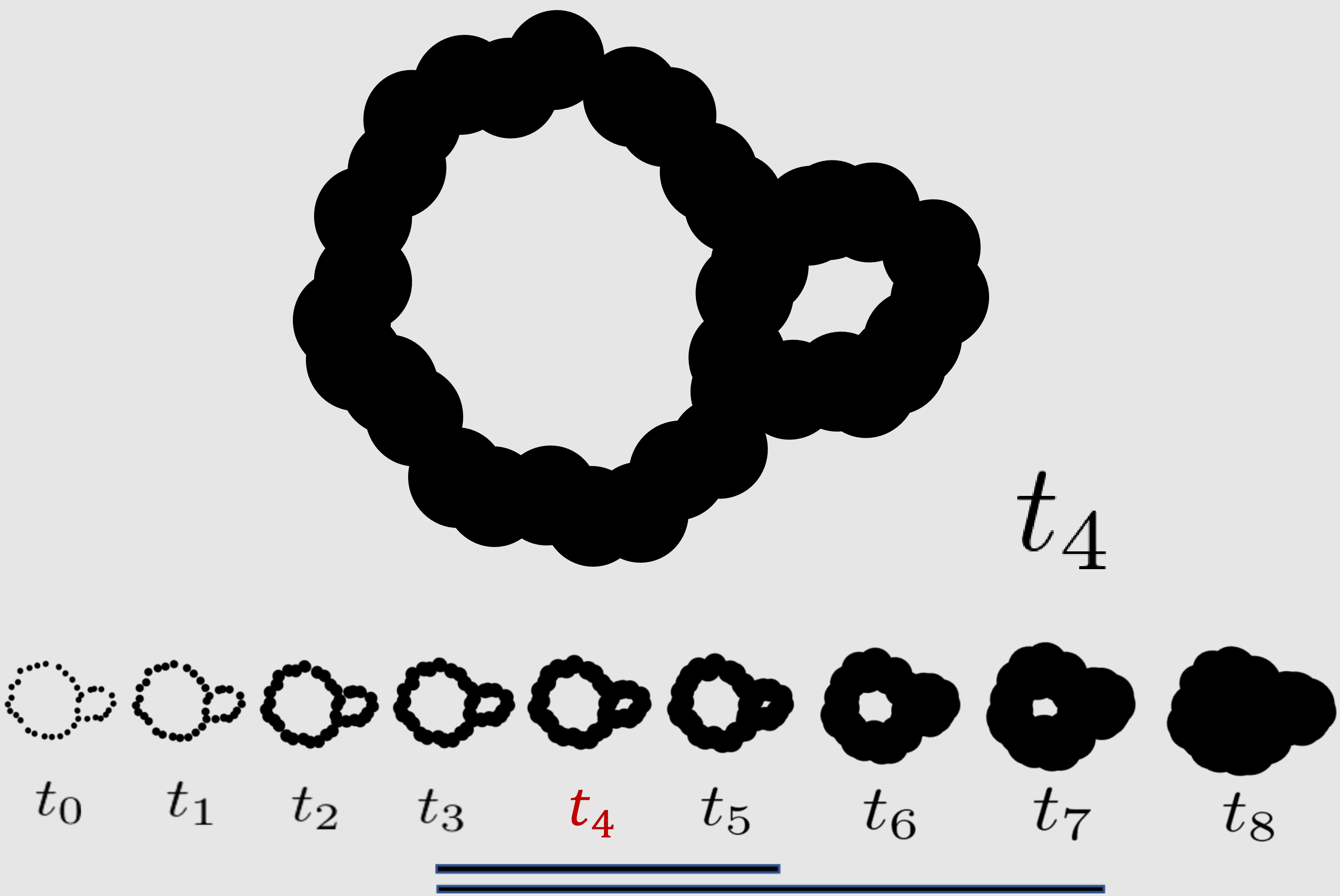

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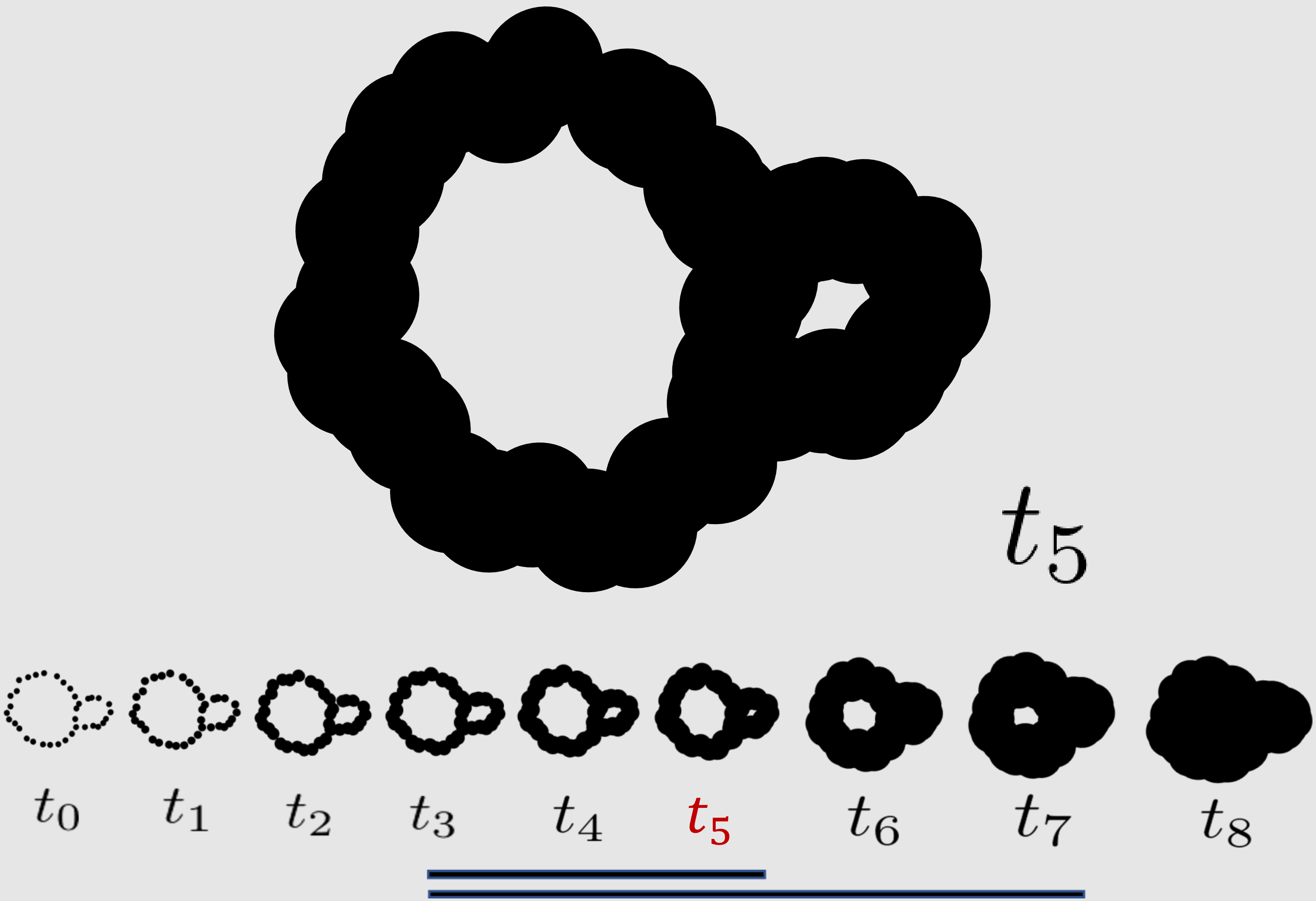

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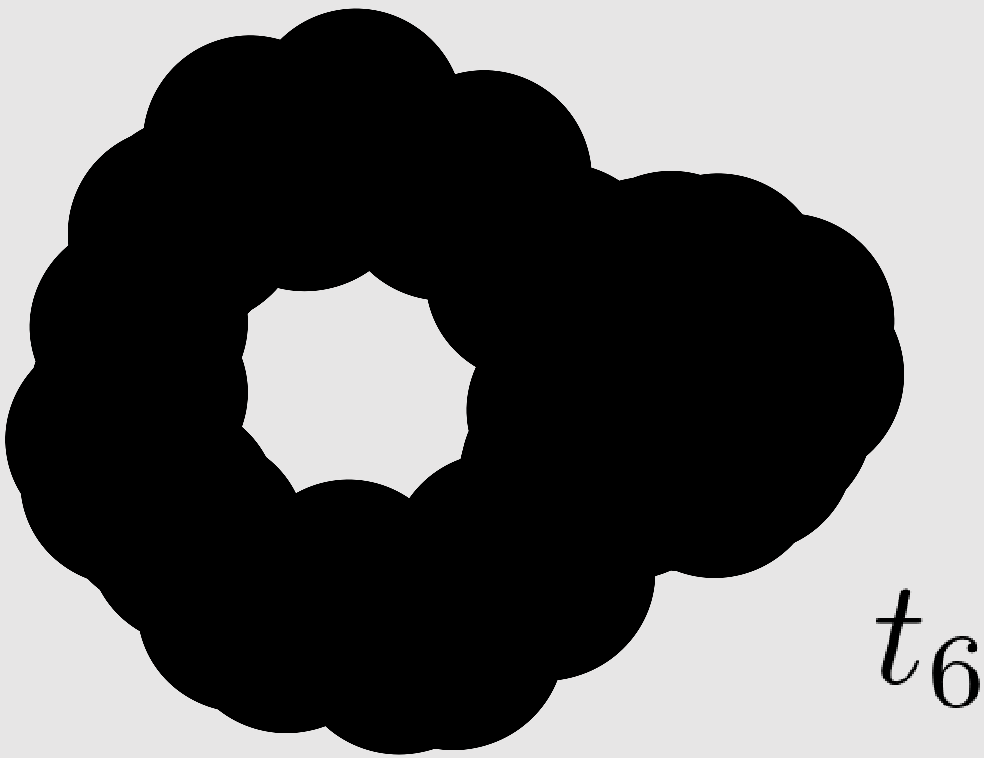

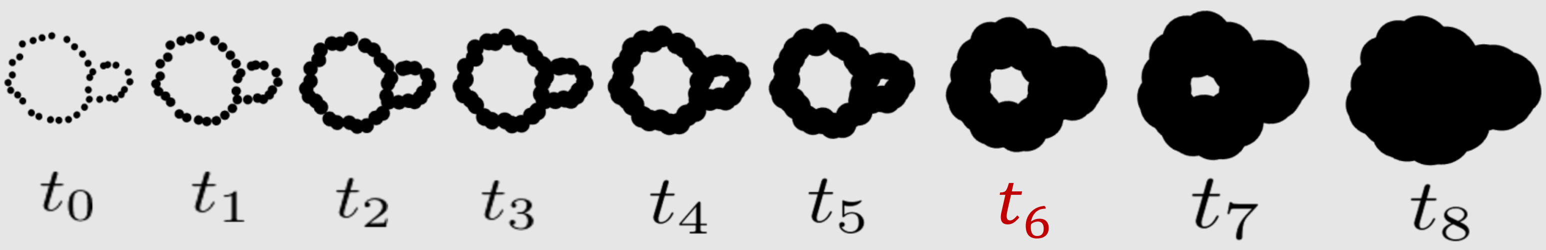

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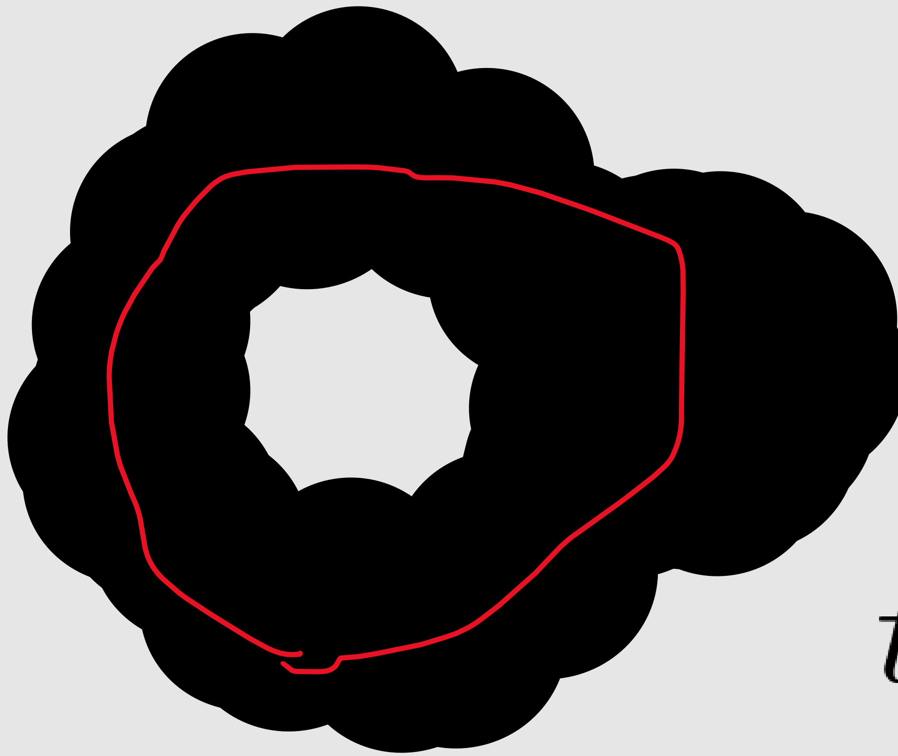
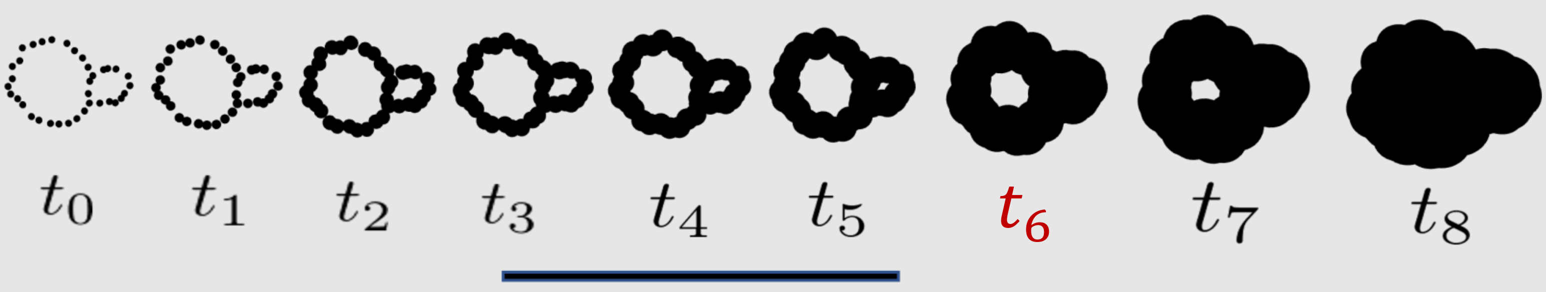
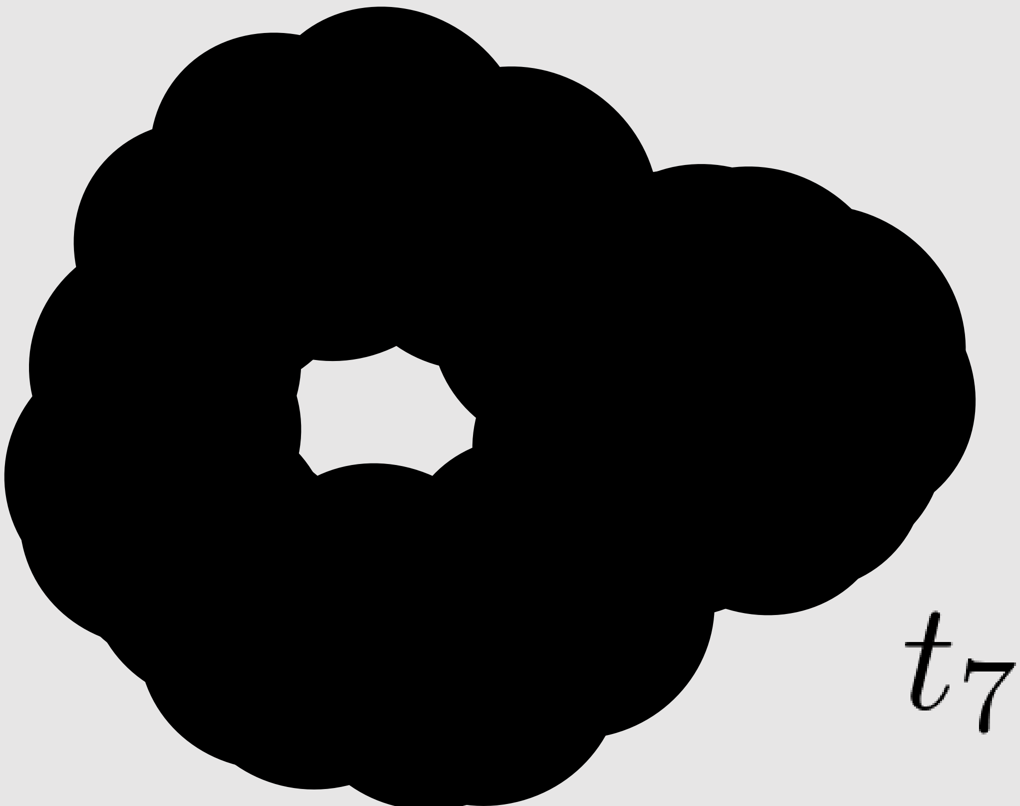


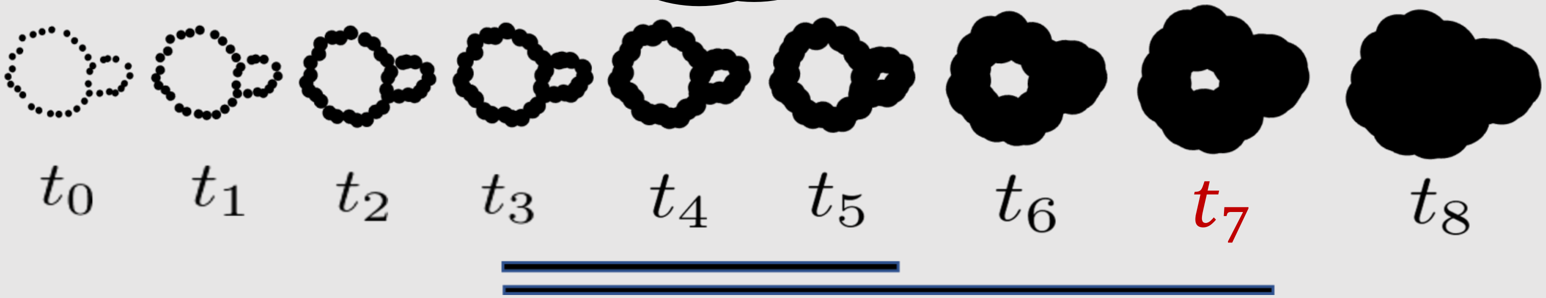
t_6 穴(右)の消滅



点群データから半径パラメータtを大きくすることにより

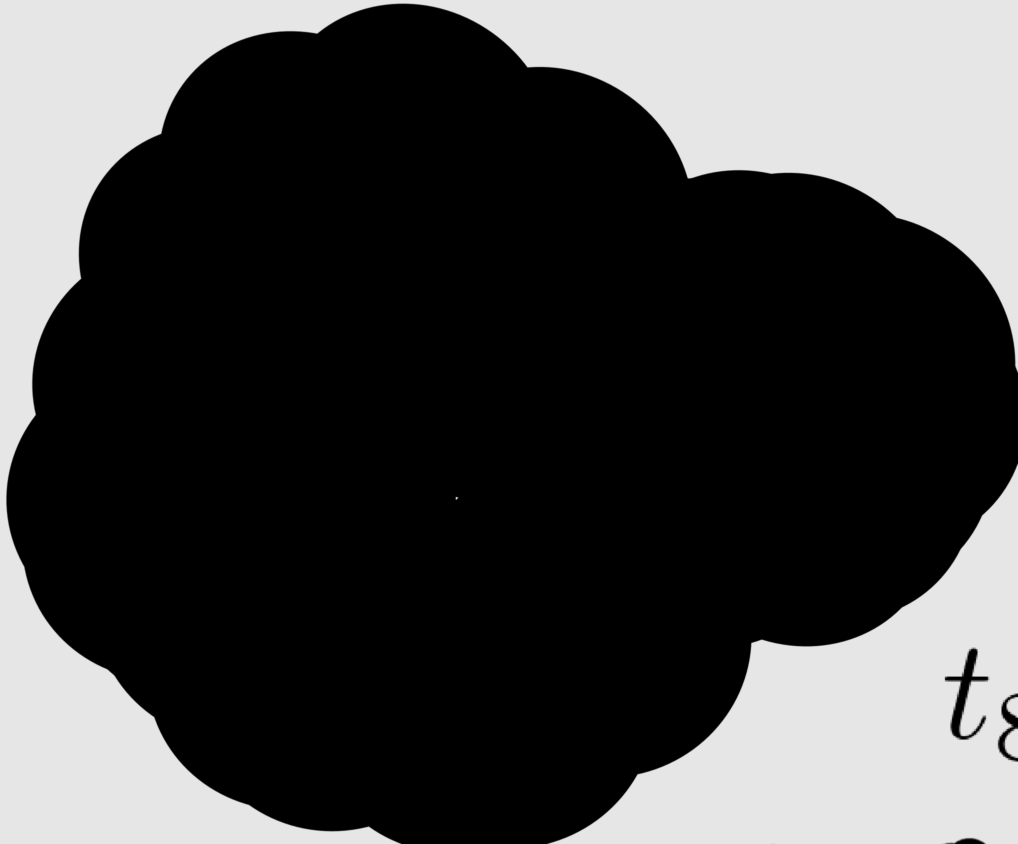
フィルトレーションを構成



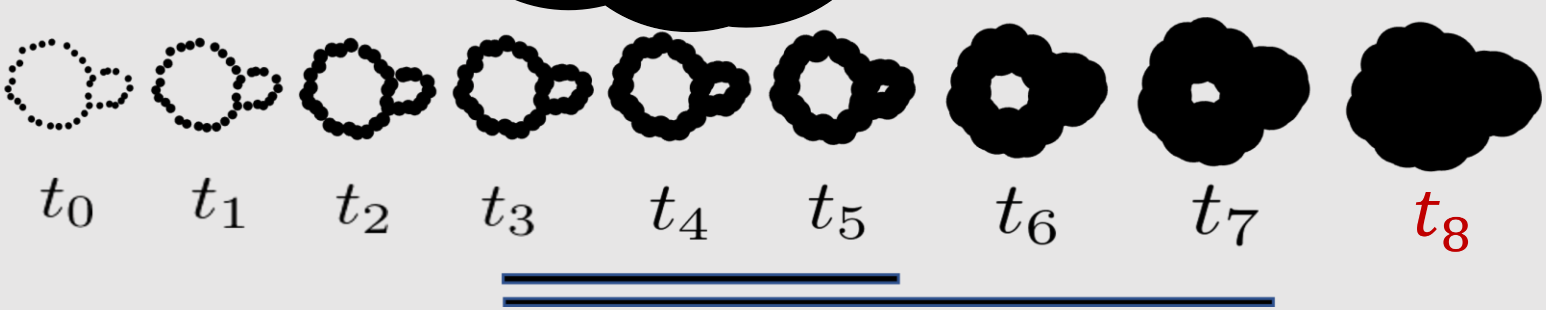


点群データから半径パラメータtを大きくすることにより

フィルトレーションを構成

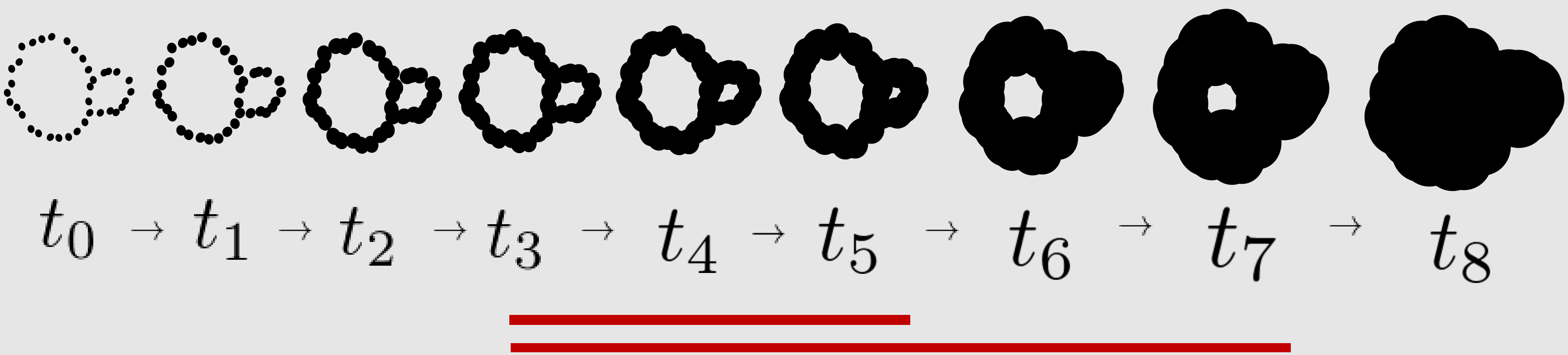


t_8 穴(左)の消滅



点群データから半径パラメータtを大きくすることにより

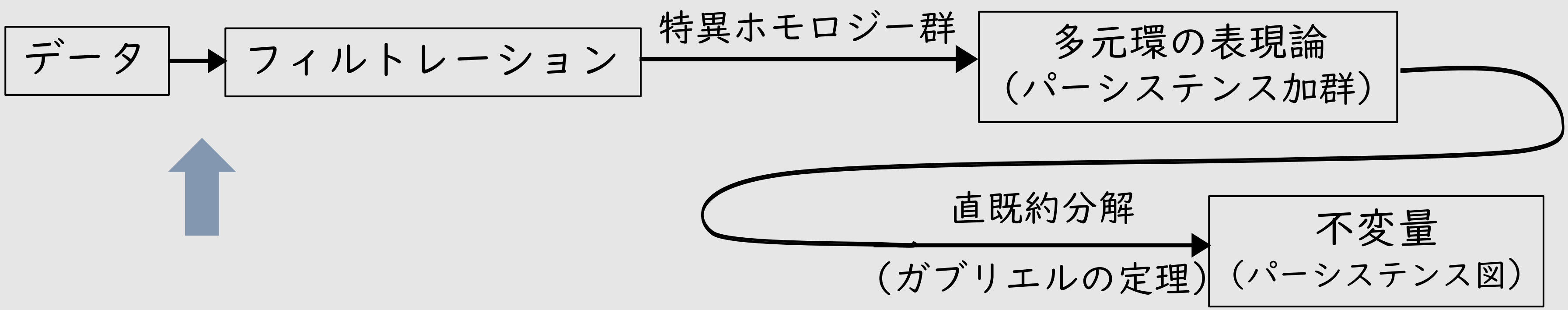
フィルトレーションを構成



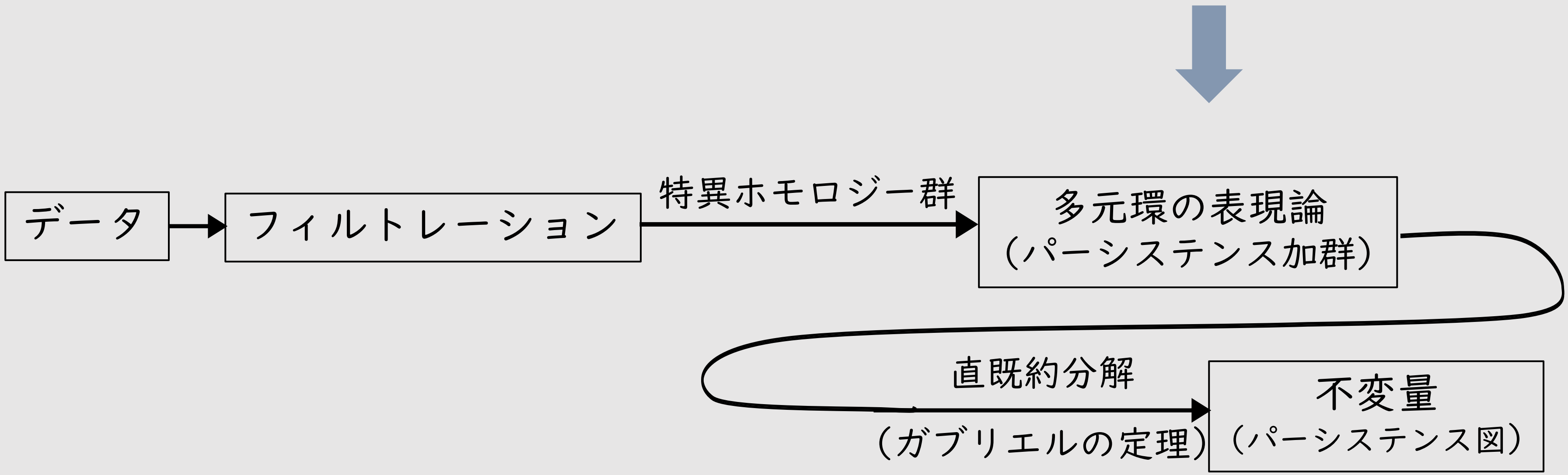
区間 [3,5], 区間 [3,7] (持続性=life-time)

によって穴の生成(birth)と消滅(death)を記述.

点群データから半径パラメータtを大きくすることにより
フィルトレーションを構成



フィルトレーション→表現論→パーシステンス図



フィルトレーション→表現論→パーシステンス図

Gabriel's theorem for type A-quivers

For a quiver

$$1 \to \cdots \to n$$

and its representations

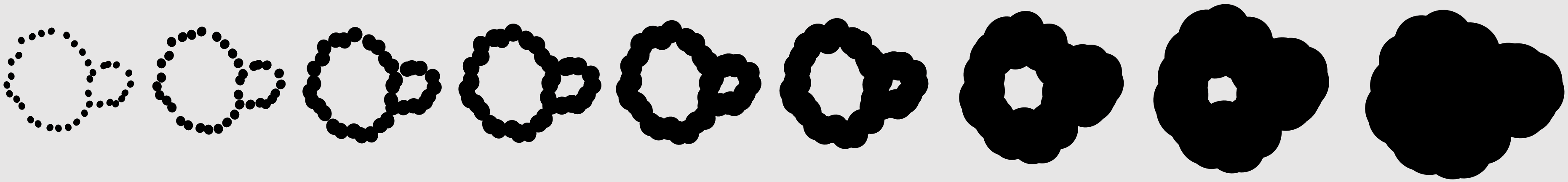
$$V : V_1 \to \cdots \to V_n,$$

we have a unique decomposition of V

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i, d_i}},$$

where $I[b_i, d_i] := \cdots \to 0 \to \underset{b_i}{k} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \underset{d_i}{k} \to 0 \to \cdots.$

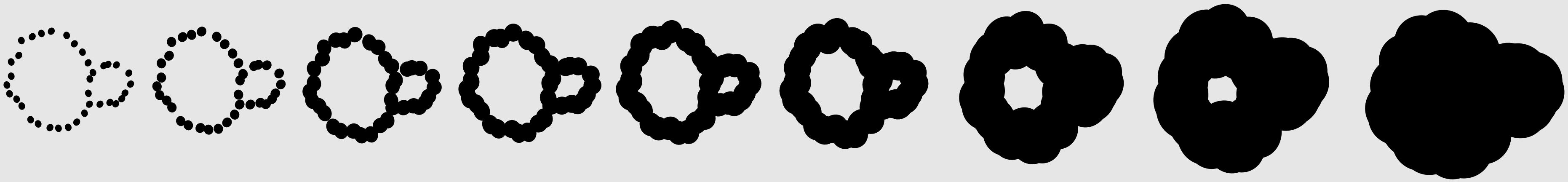
フィルトレーション→表現論→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

フィルトレーション→表現論→パーシステンス図

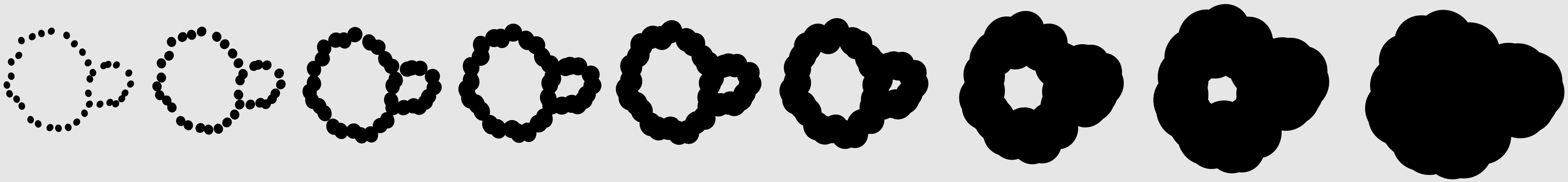


$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

フィルトレーション→表現論→パーシステンス図



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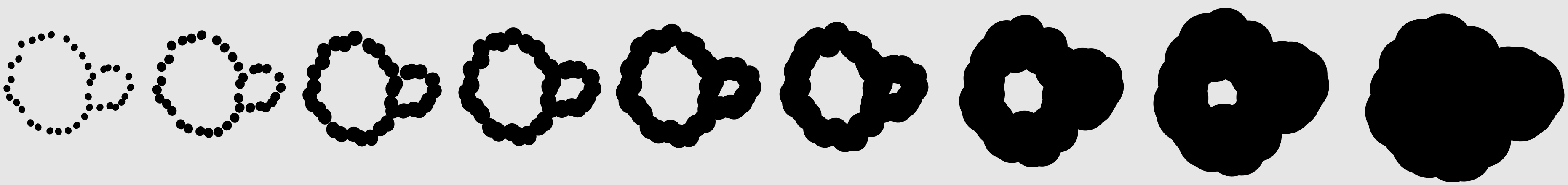
$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

フィルトレーション→表現論→パーシステンス図



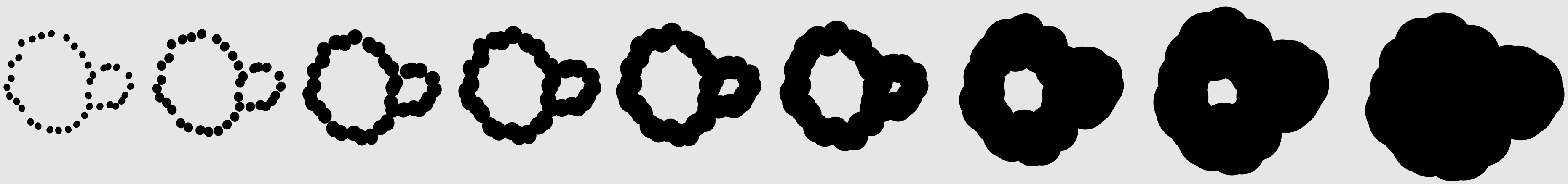
$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

$$\begin{tikzcd}[column sep=small]
H_1(t_0) \arrow[r] & H_1(t_1) \arrow[r] & H_1(t_2) \arrow[r] & H_1(t_3) \arrow[r] & H_1(t_4) \arrow[r] \arrow[d, phantom, "\cong"] & H_1(t_5) \arrow[r] & H_1(t_6) \arrow[r] & H_1(t_7) \arrow[r] & H_1(t_8) \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & k^2 \arrow[r, "\mathrm{id}"] & k^2 \arrow[r, "\mathrm{id}"] & k^2 \arrow[r, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] & k \arrow[r, "\mathrm{id}"] & k \arrow[r] & 0
\end{tikzcd}$$

A型箙の表現

フィルトレーション→表現論→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

$$\cong$$

$$0 \to 0 \to 0 \to k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\mathrm{id}} k^2 \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k \xrightarrow{\mathrm{id}} k \to 0$$

$$\cong$$

$I[3,5]$

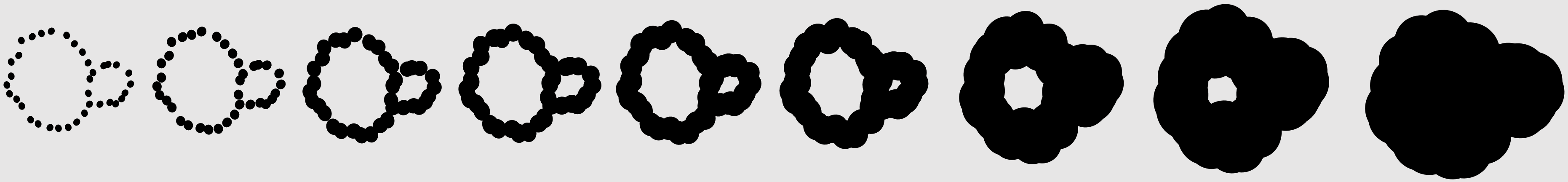
$$0 \to 0 \to 0 \to k \to k \to k \to 0 \to 0 \to 0$$

$$\oplus$$

$I[3,7]$

$$0 \to 0 \to 0 \to k \to k \to k \to k \to k \to 0$$

フィルトレーション→表現論→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

直既約加群 ⟷ 位相的特徴の持続性

$$H_1(t_0) \to H_1(t_1) \to H_1(t_2) \to H_1(t_3) \to H_1(t_4) \to H_1(t_5) \to H_1(t_6) \to H_1(t_7) \to H_1(t_8)$$

$$\cong$$

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$$\cong$$

$I[3,5]$

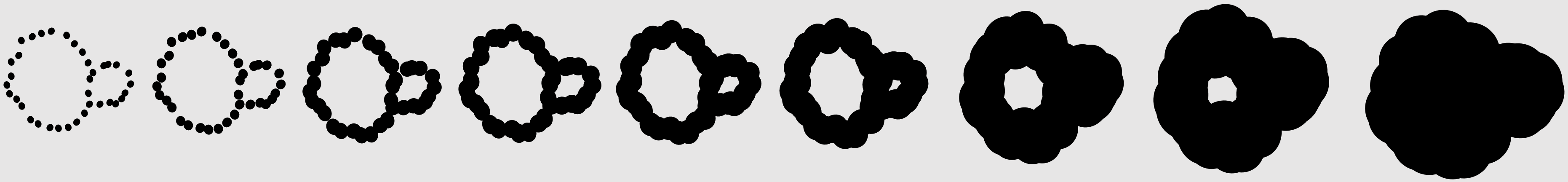
$$0 \to 0 \to 0 \to k \to k \to k \to 0 \to 0 \to 0$$

$$\oplus$$

$I[3,7]$

$$0 \to 0 \to 0 \to k \to k \to k \to k \to k \to 0$$

フィルトレーション→表現論→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$H_1(-)$

直既約加群 ⟷ 位相的特徴の持続性

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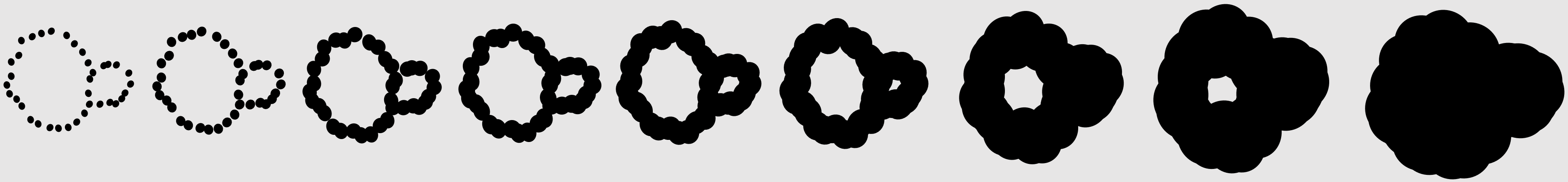
$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & \cong & & & & & & & & \\
0 & \to & 0 & \to & 0 & \to & k^2 & \xrightarrow{\mathrm{id}} & k^2 & \xrightarrow{\mathrm{id}} & k^2 & \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} & k & \xrightarrow{\mathrm{id}} & k & \to & 0 \\
 & & & & & & & & \cong & & & & & & & & I[3,5] \\
0 & \to & 0 & \to & 0 & \to & k & \to & k & \to & k & \to & 0 & \to & 0 & \to & 0 \\
 & & & & & & & & \oplus & & & & & & & & I[3,7] \\
0 & \to & 0 & \to & 0 & \to & k & \to & k & \to & k & \to & k & \to & k & \to & 0
\end{array}$$

フィルトレーション→表現論→パーシステンス図



$$t_0 \to t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7 \to t_8$$

$$\Longleftrightarrow I[3,5] \oplus I[3,7]$$

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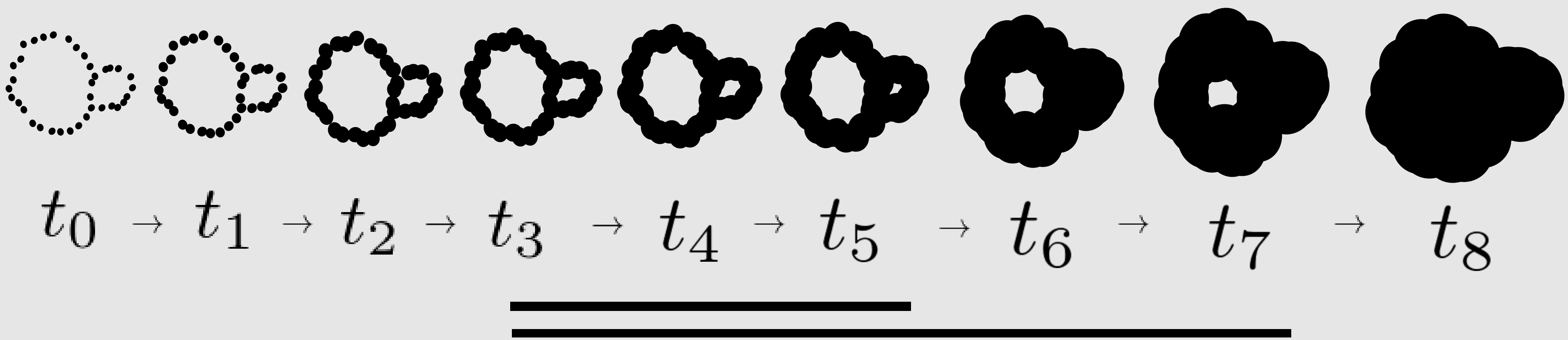
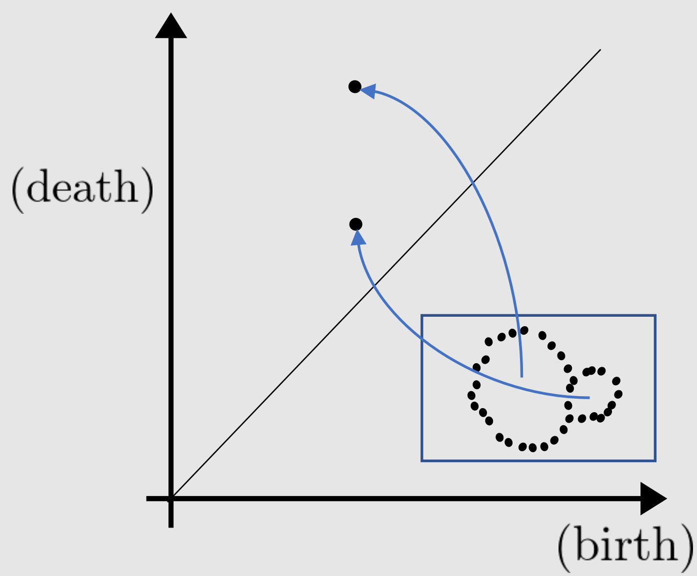


$$\Longleftrightarrow \quad I[3,5] \oplus I[3,7]$$



パーシステンス図

データの形を記述

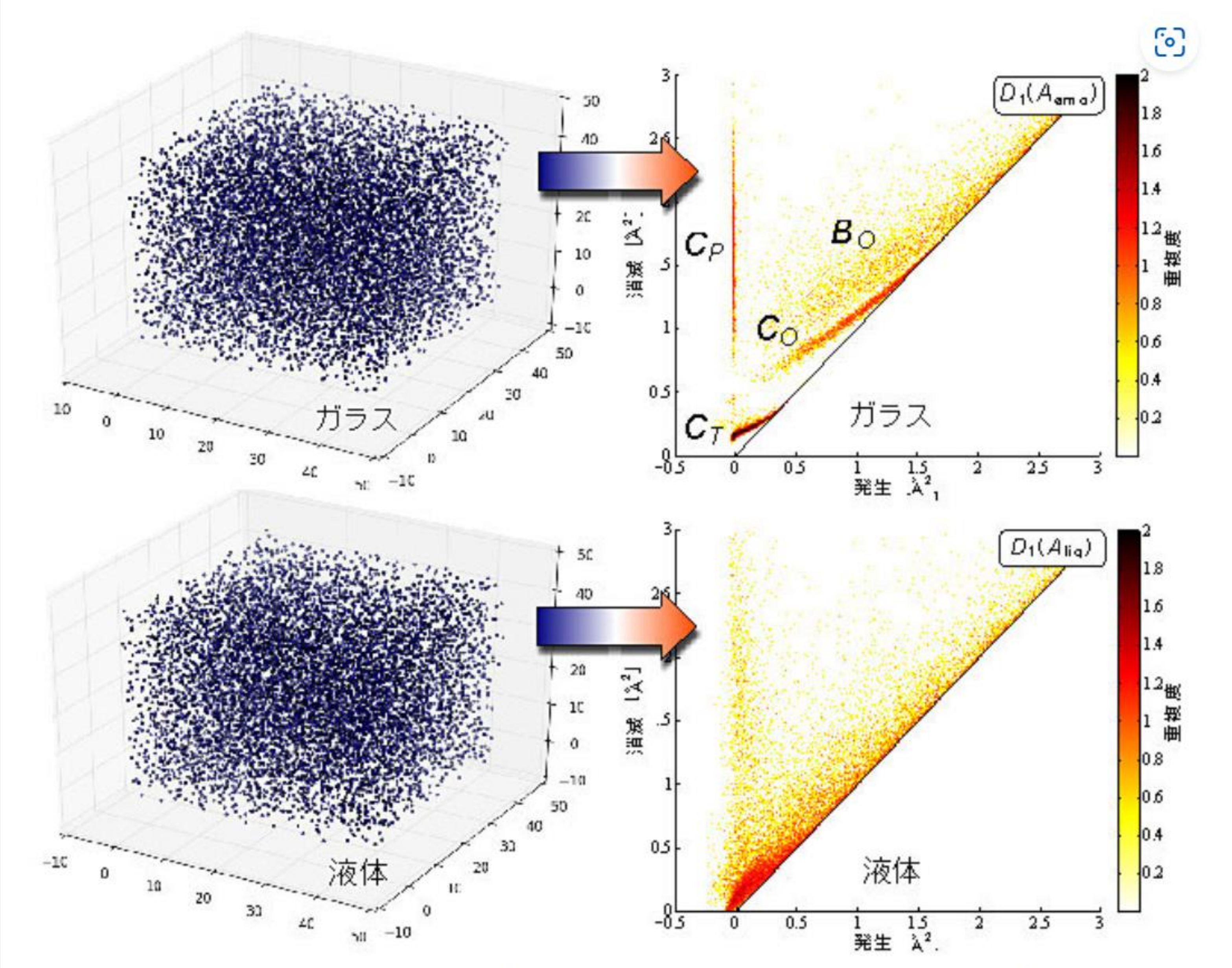
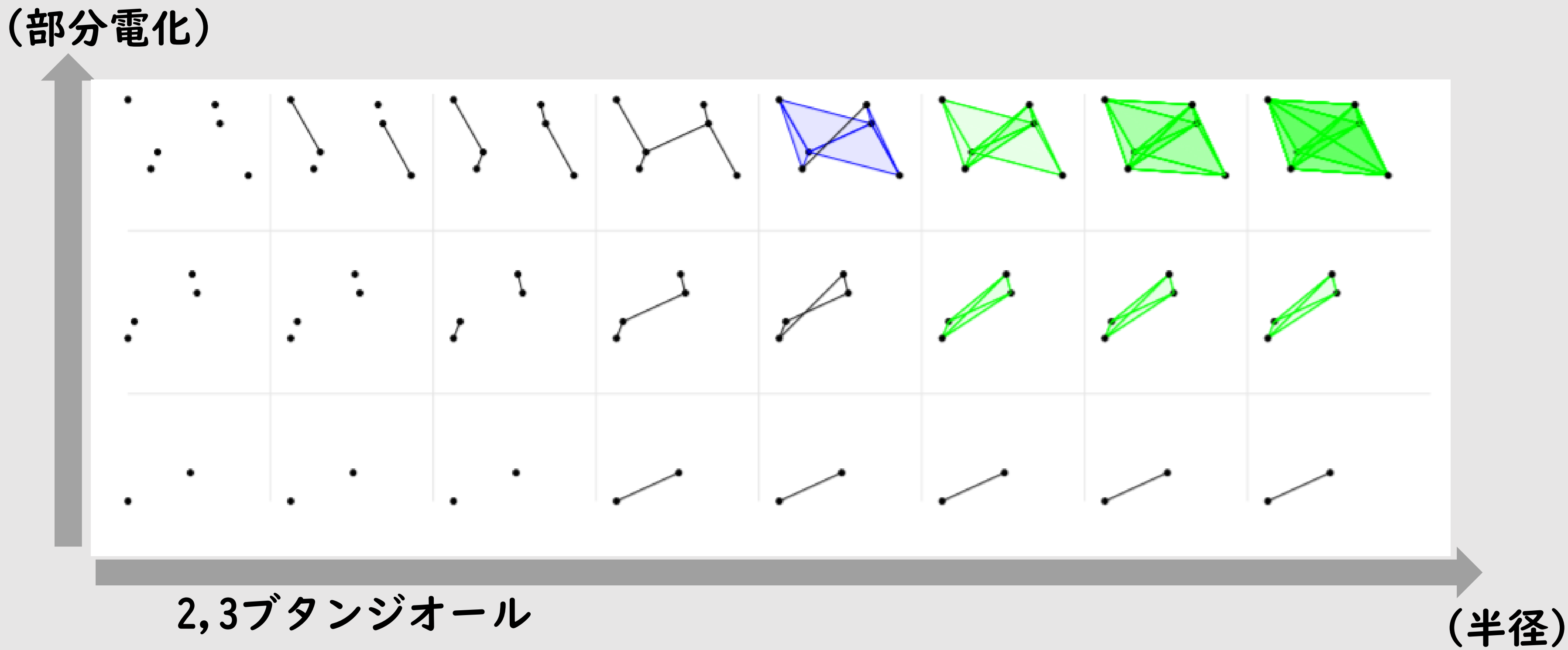


図１　ＳｉＯ２の原子配置（左）とそのパーシステントホモロジー（右）

[共同発表：ガラスの「形」を数学的に解明～トポロジーで読み解く無秩序の中の秩序～ (jst.go.jp)](https://www.jst.go.jp)**から引用**

多パラメータのパーシステントホモロジー解析



Keller B, Lesnick M, Willke TL. Persistent Homology for Virtual Screening. ChemRxiv. Cambridge: Cambridge Open Engage; 2018

多パラメータのパーシステントホモロジー解析

グリッドの表現は扱いずらい

↑ ↑ ↑ ↑ ↑ ↑

• → • → • → • → • → • →

↑ ↑ ↑ ↑ ↑ ↑

• → • → • → • → • → • →

↑ ↑ ↑ ↑ ↑ ↑

• → • → • → • → • → • →

↑ ↑ ↑ ↑ ↑ ↑

• → • → • → • → • → • →

↑ ↑ ↑ ↑ ↑ ↑

• → • → • → • → • → • →

Wild表現型

半順序集合の表現(homological algebra)

・Magnus Bakke Botnan, Steffen Oppermann, and Steve Oudot. “Signed barcodes for multi-parameter persistence via rank decompositions and rank-exact resolutions.” *In International Symposium on Computational Geometry*, 2021

・Benjamin Blanchette, Thomas Brüstle, and Eric J Hanson. “Homological approximations in persistence theory.” *Canadian Journal of Mathematics*, pages 1–38, 2021.

・Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. “Approximation by interval-decomposables and interval resolutions of persistence modules.” *Journal of Pure and Applied Algebra*, 227(10):107397, 2023.

など

区間分解可能性に着目

・→・→・→・→・ ➡ ・←→・←→・←→・←→・

一方通行からzigzagへ

・Carlsson, Gunnar, and Vin De Silva. “Zigzag persistence.” Foundations of computational mathematics 10 (2010): 367-405.

・Botnan, Magnus, and Michael Lesnick. “Algebraic stability of zigzag persistence modules.” Algebraic & geometric topology 18.6 (2018): 3133-3204.

・McDonald, R Neuhausler, R Robinson, M Larsen, L Harrington, H Bruna, M “Zigzag persistence for coral reef resilience using a stochastic spatial model.” Journal of the Royal Society, Interface volume 20 issue 205 20230280-(23 Aug 2023).

発表の流れ

■位相的データ解析とは？

■定理(常に区間分解可能な半順序集合)

■定理(2)

Persistence module (1/2)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leqq b \Leftrightarrow \exists !\ a \to b$)

-

Persistence module (1/2)

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- *Persistence modules over P* are functors from P to k-mod

  (or equivalently modules over incidence algebra $k[P]$).

Intervals (2/2)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).
-
-

$$I \subset P: \quad \begin{tikzcd}[column sep=small, row sep=small]
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd} \qquad k_I: \quad \begin{tikzcd}[column sep=small, row sep=small]
k \arrow[dr] & & k \arrow[dr] & & 0 \\
& k \arrow[ur] \arrow[dr] & \circlearrowleft & k \arrow[ur] & \\
& & k \arrow[ur] \arrow[dr] & & \\
& & & k &
\end{tikzcd}$$

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$$\begin{tikzcd}
\bullet \arrow[dr] & & \circ \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd}$$

$$I \subset P: \quad \begin{tikzcd}
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd} \qquad k_I: \quad \begin{tikzcd}
k \arrow[dr] & & k \arrow[dr] & & 0 \\
& k \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & k \arrow[ur] & \\
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& & & k &
\end{tikzcd}$$

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- For an interval I of P, the *interval module k_I* is defined by
  $k_I(p) \coloneqq k$ for $p \in I$, otherwise $k_I(p) \coloneqq 0$,
  $k_I(a \to b) \coloneqq \mathrm{id}_k$ for $a, b \in I$, otherwise 0.

-

$$I \subset P : \quad \begin{tikzcd}
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd}
\qquad\qquad
k_I : \quad \begin{tikzcd}
k \arrow[dr] & & k \arrow[dr] & & 0 \\
& k \arrow[ur] \arrow[dr] & \circlearrowleft & k \arrow[ur] & \\
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& & & k &
\end{tikzcd}$$

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- A module is *interval decomposable* if the module decomposes into interval modules.

$I \subset P$:

$$
\begin{tikzcd}
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd}
$$

k_I :

$$
\begin{tikzcd}
k \arrow[dr] & & k \arrow[dr] & & 0 \\
& k \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & k \arrow[ur] & \\
& & k \arrow[ur] \arrow[dr] & & \\
& & & k &
\end{tikzcd}
$$

Theorem 1 [Aoki-Escolar-T]

Let P be a finite poset. The following are equivalent.

(a) Every $k[P]$ module is interval decomposable.
(b) The Hasse diagram of P is one of the following form:

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

A_n (a)

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & \arrow[r, phantom, "\circlearrowleft"] & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\tilde{A}_{n,m}$

発表の流れ

■位相的データ解析とは？

■定理(常に区間分解可能な半順序集合)

■定理(区間次元)

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n\,(a)$

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & \circlearrowleft & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\tilde{A}_{n,m}$

（加群が常に区間分解可能な半順序集合）

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n\,(a)$

$$\begin{array}{ccccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & \circlearrowleft & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

$\tilde{A}_{n,m}$

（加群が常に区間分解可能な半順序集合）

一般の半順序集合の表現

$$\begin{array}{cccccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
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\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to
\end{array}$$

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n\,(a)$

$$\begin{tikzcd}[row sep=small]
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & \arrow[phantom, "\circlearrowleft"] & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\tilde{A}_{n,m}$

（加群が常に区間分解可能な半順序集合）

⇒ 一般の半順序集合の表現

半順序集合上の（データから得られるような）
複雑な加群を、

“取り扱いやすい加群”（区間加群）

を用いて理解したい.

$$\begin{array}{cccccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
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\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to
\end{array}$$

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n\,(a)$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & \circlearrowleft & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

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（加群が常に区間分解可能な半順序集合）

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\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
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\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow
\end{array}$$

区間近似
(Interval approximation)

Resolution dimension (1/2)

- A : a finite dimensional k-algbra
- $\mathcal{X}$: a full subcategory of mod A satisfying certain conditions.
  (proj(A) $\subseteq \mathcal{X}$, functorial finite, closed under direct summand, …)
  e.g. $\mathcal{X}$ = the set of all interval decomposable modules.

- If M has a right minimal $\mathcal{X}$-resolution of the form

$$0 \to J_m \xrightarrow{g_m} \cdots \to J_2 \xrightarrow{g_2} J_1 \xrightarrow{g_1} J \xrightarrow{f} M \to 0,$$

then we say that the $\mathcal{X}$-*resolution dimension of M* is m and write $\mathcal{X}$-res-dim $M = m$.

Otherwise, we say that the $\mathcal{X}$-resolution dimension of M is infinity.

Interval resolution global dimension (2/2)

Now, we consider

- $k[P]$: the incidence algebra of a poset P.
- $\mathcal{J}_P$: the set of interval decomposable modules over $k[P]$.
- For a module M, let int-res-dim(M) be the resolution dimension of M with respect to $\mathcal{J}_P$.
-

Interval resolution global dimension (5/5)

Now, we consider

- $k[P]$: the incidence algebra of a poset P.
- $\mathcal{I}_P$: the set of interval decomposable modules over $k[P]$.
- For a module M, let int-res-dim(M) be the resolution dimension of M with respect to $\mathcal{I}_P$.

- *interval resolution global dimension of* $k[P]$ is

$$\text{int-res-gldim}(k[P]) := \sup\{ \text{ int-res-dim}(M) \mid M \in \text{mod } k[P] \}$$

($< \infty$, by [Asashiba-Escolar-Nakashima-Yoshiwaki, 23])

Interval resolution global dimension (5/5)

Now, we consider

- $k[P]$: the incidence algebra of a poset P.
- $\mathcal{I}_P$: the set of interval decomposable modules over $k[P]$.
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- *interval resolution global dimension of* $k[P]$ is

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($< \infty$, by [Asashiba-Escolar-Nakashima-Yoshiwaki, 23])

Remark

int-res-gldim($k[P]$) is zero if and only if the Hasse diagram of P is either (i) or (ii), where

(i) $\underset{A_n\,(a)}{1 \longleftrightarrow \cdots \longleftrightarrow n}$, (ii)
$$\begin{tikzcd}[row sep=small, column sep=small]
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & \circlearrowleft & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$
.

Theorem 2 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}\ k[Q] \leq \text{int-res-gldim}\ k[P].$$

Remark

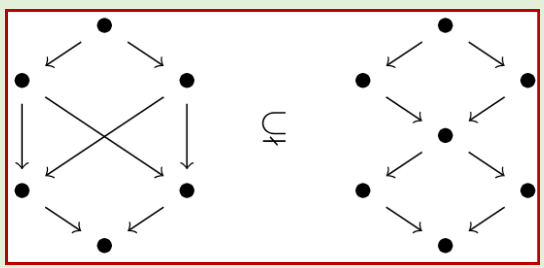
Theorem 2 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}\ k[Q] \leq \text{int-res-gldim}\ k[P].$$

Remark

The above monotonicity does not hold for (usual) global dimension in general [Igusa-Zacharia, 1990].



Poset	Q	$\subset$	P	
Global dimension	3	>	2	
Interval global dimension	1	<	2	(over a field with two elements)

For a full subposet Q of P, we have an isomorphism

$$k[Q] \cong ek[P]e$$

of k-algebras, where $e \coloneqq \Sigma_{x \in Q} e_x$. It induces adjoint functors

$$\begin{tikzcd} \operatorname{mod} k[P] \arrow[rr, "\mathrm{Res}"] & & \operatorname{mod} k[Q] \arrow[ll, bend right, "\mathrm{T} \coloneqq - \otimes_{k[Q]} ek[P]"'] \arrow[ll, bend left, "\mathrm{L} \coloneqq \mathrm{Hom}_{k[Q]}(ek[P], -)"] \end{tikzcd}$$

- Res preserves interval decomposability of modules.
- T and L do NOT preserve interval decomposability of modules in general.

We find a functor Θ that sends to interval modules over Q to interval modules over P by using T and L.

The functor Θ

Using adjoint functors, we have

$$\begin{array}{ccc} \mathrm{Hom}_{k[Q]}(M,M) & \cong & \mathrm{Hom}_{k[P]}\big(\mathrm{T}(M),\mathrm{L}(M)\big). \\ \Cup & & \Cup \\ 1_M & \longmapsto & \theta_M \end{array}$$

For a given module $M \in \mathrm{mod}\, k[Q]$, let

$$\Theta(M) \coloneqq \mathrm{Im}(\theta_M).$$

It gives rise to a functor Θ. It is called *intermediate extension* in [Kuhn, 94], and *prolongedment intermédiare* in [Beilison-Bernstein-Deligne, 82].

Proposition For a given interval I of Q, let k_I be the corresponding interval $k[Q]$-module. Then, we have

$$\Theta(k_I) \cong k_{\mathrm{conv}(I)},$$

where $\mathrm{conv}(I)$ is the smallest interval of P containing I.

The functor Θ

We obtain a pair of functors

$$\begin{tikzcd} \operatorname{mod} k[P] \arrow[r, bend right, "\mathrm{Res}"'] & \operatorname{mod} k[Q] \arrow[l, bend right, "\Theta"'] \end{tikzcd}$$

satisfying the following properties :

(i) Res preserves interval decomposability of modules.
(ii) Θ sends interval modules to interval modules by Proposition.
(iii) $1_{\operatorname{mod} k[Q]} \cong \mathrm{Res} \circ \Theta$.

Proposition For any $M \in \operatorname{mod} k[Q]$, we have the following inequality

$$\text{int-res-dim}(M) \leq \text{int-res-dim}\big(\Theta(M)\big).$$

Since M is an arbitrary module, we obtain the desired inequality

$$\text{int-res-gldim}(k[Q]) \leq \text{int-res-gldim}(k[P]). \qquad \square$$

Discussion

- Can we apply $\tilde{A}_{n,m}$ to topological data analysis? Stability?
- Does int-res-gldim depend on the characteristic of fields?
- Computation using GAP package QPA(“Quiver and Path Algebras”)

  -pmgap : $(n \times m)$-grid by E. G. Escolar

  -our project: an arbitrary finite poset
  e.g. interval approximations / resolutions of modules

GAP - Groups, Algorithm, Programming -
a System for Computational Discreate Algebra

ご清聴ありがとうございました。

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