

Aligned Interior Systems and Induced Functors for Persistence Modules

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T. Aoki, S. Tada, On preservation of relative resolutions for poset representations,
arXiv:2506.21227 (2025).

Goal: Galois connections induce functors between categories of persistence modules. We introduce the **alignment conditions** on Galois connections and study the induced functors for (homological) invariants.

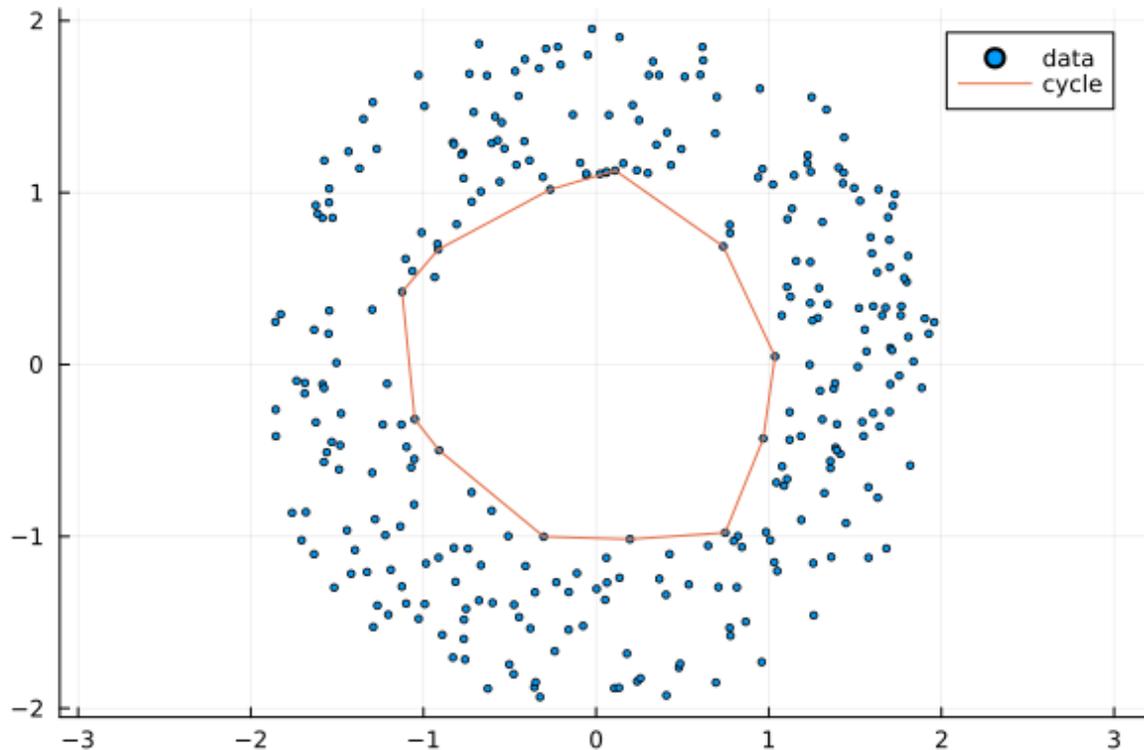
$$\begin{array}{ccc}
 \begin{array}{c}
 Q \begin{array}{c} \xrightarrow{l_Q} \\ \perp \\ \xleftarrow{\lfloor \rfloor_Q} \end{array} P \\
 \text{Galois connection}
 \end{array}
 & \rightsquigarrow &
 \begin{array}{c}
 \text{rep}_k(Q) \begin{array}{c} \xleftarrow{\text{Ind}_Q} \\ \text{Ind}_Q := \lfloor \rfloor_Q^* \\ \xrightarrow{\text{Cont}_Q} \end{array} \text{rep}_k(P) \\
 \text{Res}_Q := l_Q^* \\
 \text{Cont}_Q
 \end{array}
 \end{array}$$

Contents

- (1) Introduction: (Multiparameter) persistent homology
- (2) Preliminary
- (3) Induced functors by (aligned) interior systems
- (4) Applications

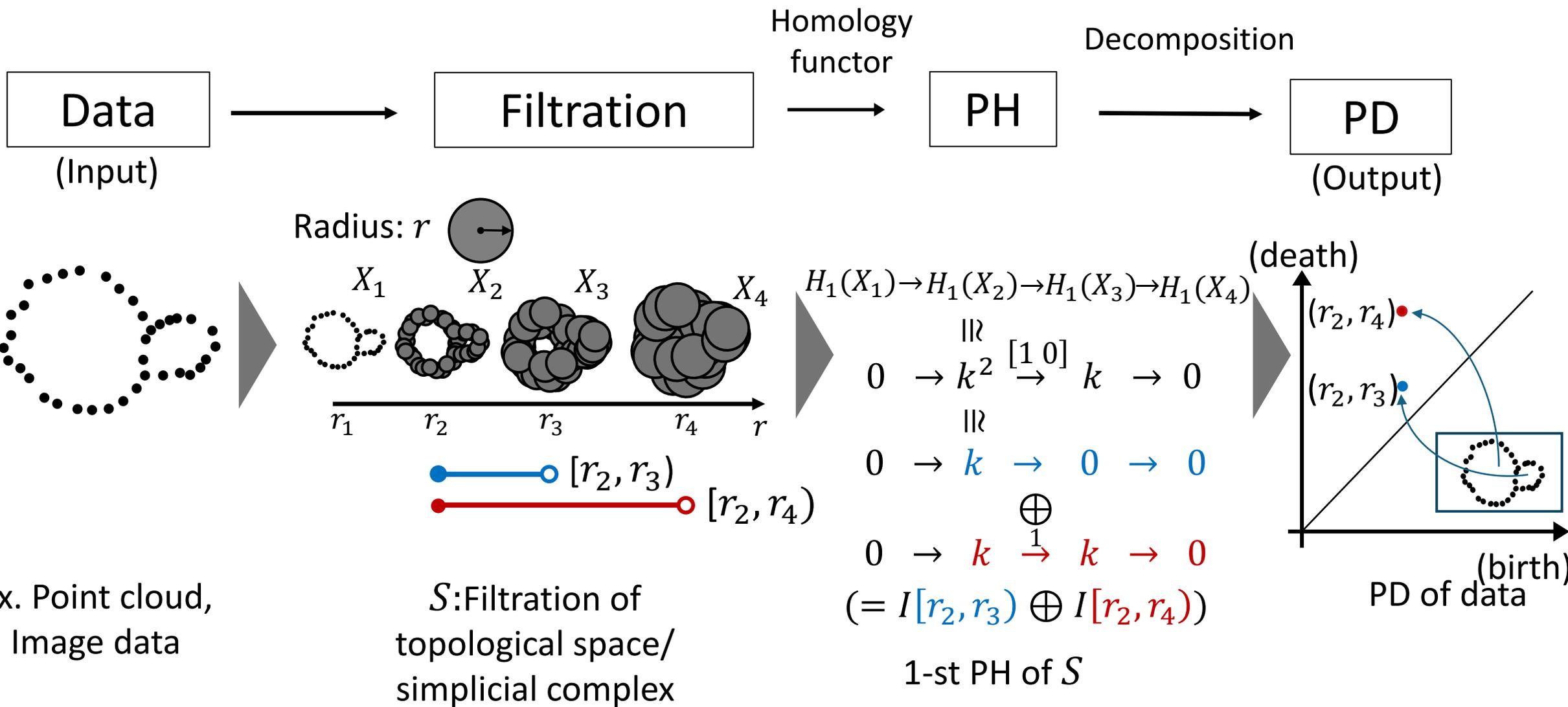
Introduction

- *Persistent homology* (PH) is a tool in Topological Data Analysis.
- It captures the persistence of “shape” (connected components, holes or voids) of data by a *persistence diagram* (PD).



Introduction

PH: Persistent homology
 PD: Persistence diagram



Introduction: applications

- Material science
- Evolutional biology
- Computational gastronomy
and others...

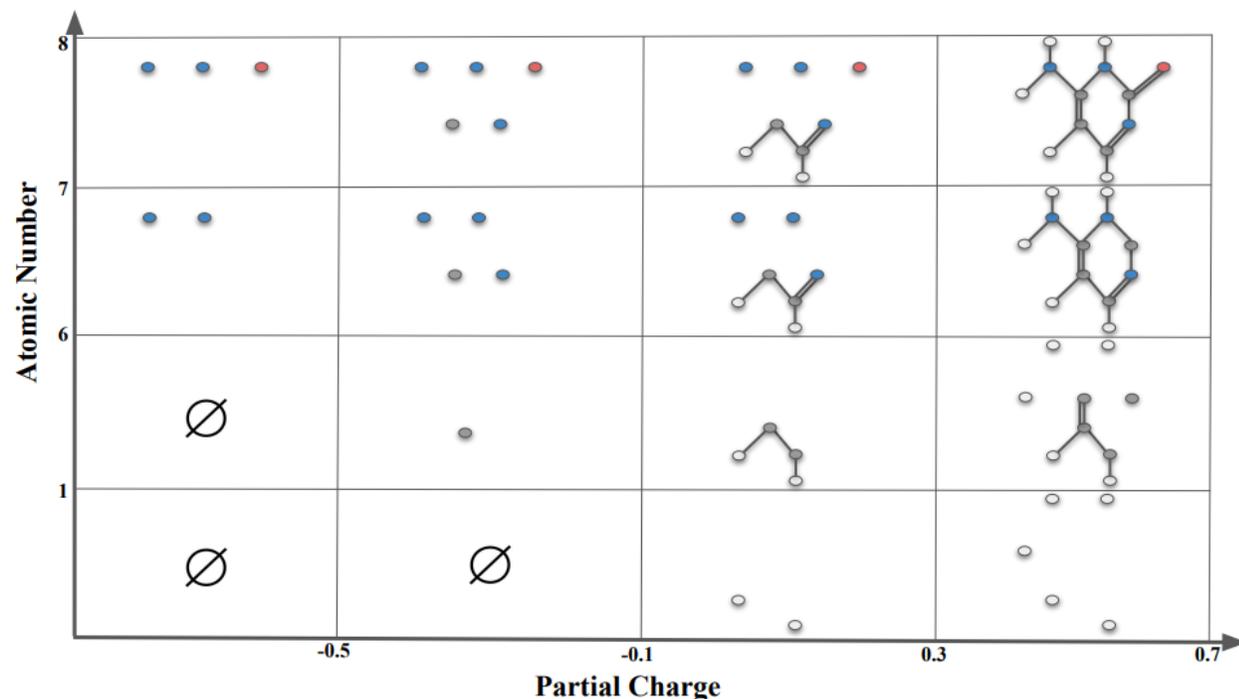
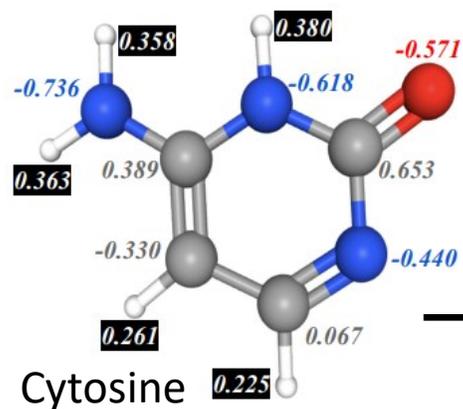
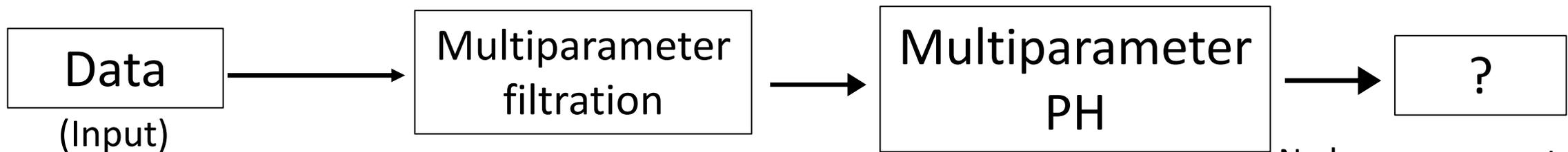
• Yasuaki Hiraoka, Takenobu Nakamura, Akihiko Hirata, Emerson G. Escolar, Kaname Matsue, and Yasumasa Nishiura. Hierarchical structures of amorphous solids characterized by persistent homology. *Proceedings of the National Academy of Sciences*, 113(26), 7035-7040, 2016.

• Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. Topology of viral evolution. *Proceedings of the National Academy of Sciences*, 110(46):18566–18571, 2013

• Emerson G. Escolar, Yuta Shimada, and Masahiro Yuasa. A topological analysis of the space of recipes. *International Journal of Gastronomy and Food Science*, 39:101088, 2025.

Introduction (MPH)

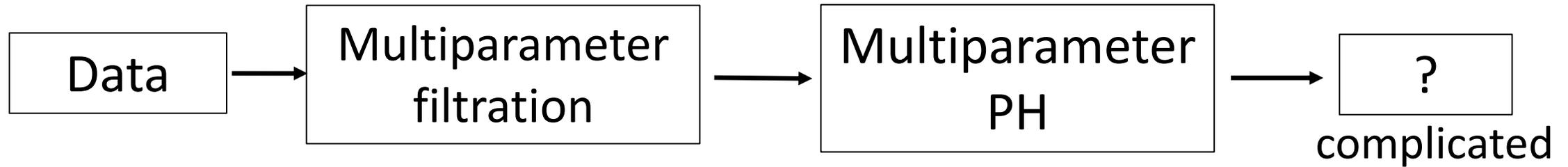
Multiparameter Persistent Homology arises from multiparameter data.



No longer parametrized by intervals (complicated)

Figures: Demir, Andac, et al. "ToDD: Topological compound fingerprinting in computer-aided drug discovery". *Advances in Neural Information Processing Systems* 35 (2022): 27978-27993.

Introduction (MPH)



Use of intervals (which play a central role in PH), for example...

- The generalized rank invariant,
- The fibered barcode,
- **The homological invariant relative to intervals (interval approximations).**

• Nathaniel Clause and Woojin Kim and Facundo Mémoli. The Generalized Rank Invariant: Möbius invertibility, Discriminating Power, and Connection to Other Invariants. arXiv:2207.11591, 2024.

• Michael Lesnick and Matthew Wright. Interactive Visualization of 2-D Persistence Modules. arXiv:1512.00180, 2015.

• Benjamin Blanchette, Thomas Brüstle, and Eric J. Hanson. Homological approximations in persistence theory. *Canadian Journal of Mathematics*, pages 1–38, 2021.

• Hideto Asashiba, Emerson G. Escolar, Ken Nakashima, and Michio Yoshiwaki. Approximation by interval-decomposables and interval resolutions of persistence modules. *Journal of Pure and Applied Algebra*, 227(10):107397, 2023.

Contents

(1) Introduction: (Multiparameter) persistent homology

(2) **Preliminary**

(3) Induced functors by (aliened) interior systems

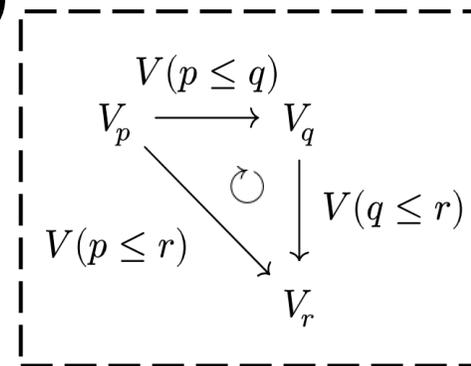
(4) Applications

Persistence modules

- Let k be a field.
- Let P be a finite poset (as a category $x \leq y \Leftrightarrow \exists x \rightarrow y$).
- A P -persistence module is an object in $\text{rep}_k(P) := \text{Fun}(P, \text{vect}_k)$.

Equivalently, it consists of data $V = \left(V_p, V(p \leq q) \right)_{p \leq q \in P}$ s. t.

- V_p is a fin. dim. k -vector space ($\forall p \in P$).
- $V(p \leq q): V_p \rightarrow V_q$ is a linear map satisfying
$$V(p \leq p) = \text{id}_{V_p} \quad (\forall p \in P).$$
$$V(p \leq r) = V(q \leq r) \circ V(p \leq q) \quad (\forall p \leq q \leq r \in P)$$



Interval module

- Let I be a subset of P . I is an **interval** if it is
 - (1) convex ($a \leq c \leq b$ and $a, b \in I$ imply $c \in I$).
 - (2) connected ($\forall a, b \in I, \exists a = z_0, z_1, \dots, z_n = b$ in I s. t., z_i and z_{i+1} are comparable for any $i \in \{0, \dots, n - 1\}$).

• A P -persistence module is said to be an **interval module** if it is isomorphic to $k_I = ((k_I)_a, k_I(a \leq b))_{\{a \leq b \in P\}}$, where

$$(k_I)_a := k \text{ for } a \in I, \text{ otherwise } (k_I)_a := 0,$$
$$k_I(a \leq b) := \text{id}_k \text{ for } a \leq b \in I, \text{ otherwise } 0,$$

for an interval I .

• A persistence module is **interval-decomposable** if it decomposes into interval modules.

Interval approximation/cover

- \mathcal{I}_P : the set of interval-decomposable P -persistence modules.
- V : a P -persistence module.

An **interval approximation** of V is a morphism $f: J \rightarrow V$ with $J \in \mathcal{I}_P$ s. t. every morphism $g: I \rightarrow V$ with $I \in \mathcal{I}_P$ factors through f .

$$\begin{array}{ccc} \mathcal{I}_P \ni I & & \\ \downarrow \exists & \searrow \forall g & \\ \mathcal{I}_P \ni J & \xrightarrow{f} & V \end{array}$$

An **interval cover** of V is an interval approximation such that the number of direct summands of the domain is smaller than equal to that of other interval approximations (uniquely determined).

Interval resolution dimension

- An *interval resolution* of $V \in \text{rep}_k(P)$ is an exact sequence

$$\begin{array}{ccccccc} \cdots & \rightarrow & J_m & \xrightarrow{g_m} & \cdots & \rightarrow & J_2 & \xrightarrow{g_2} & J_1 & \xrightarrow{g_1} & J & \xrightarrow{f} & V & \rightarrow & 0. \\ & & & & & & \nearrow^{l_3} & \searrow_{f_2} & \nearrow^{l_2} & \searrow_{f_1} & \nearrow^{l_1} & & & & \\ & & & & \cdots & & K_3 & & K_2 & & K_1 & & & & \\ & & & & & & & & & & & & f: \text{interval cover of } V. & & \\ & & & & & & & & & & & & f_1: \text{interval cover of } K_1. & & \\ & & & & & & & & & & & & \cdots & & \end{array}$$

- We say that the *interval resolution dimension* of V is m if $J_m \neq 0$ and $J_{m+1} = 0$, and write

$$\text{int-res-dim}(V) = m.$$

Interval resolution global dimension

- The *interval resolution global dimension* of P is defined by

$$\text{int-res-gldim}(P) := \sup\{\text{int-res-dim}(V) \mid V \in \text{rep}_k(P)\}.$$

Proposition [AENY, '23, Props. 4.5, 4.6]

Let P be a finite poset. We have the equalities among all the following values:

(1) $\text{int-res-gldim}(P)$

(2) $\sup\{\text{int-res-dim } \tau k_I \mid I: \text{an interval in } P\}.$

(3) $\sup\{\text{int-res-dim } \Gamma k_I \mid I: \text{an interval in } P\}.$

Moreover, the value is finite.

- τ is the Auslander-Reiten translation.
- $\Gamma k_I := \text{Ker}(\bigoplus_{k_J \rightarrow k_I: \text{relative irreducible}} k_J \rightarrow k_I)$

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- Interior systems

- Induced functors

- Alienated interior systems

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Interior system

Definition

Let Q be a full subposet of P . We say that Q is an *interior system* of P if $\iota_Q: Q \hookrightarrow P$ has a right adjoint.

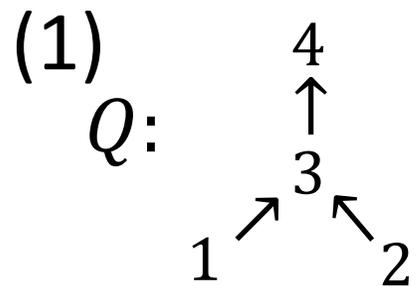
- $Q \subseteq P$: an interior system. The floor function

$$\lfloor \cdot \rfloor_Q: P \rightarrow Q, \quad p \mapsto \max\{q \in Q \mid q \leq p\}$$

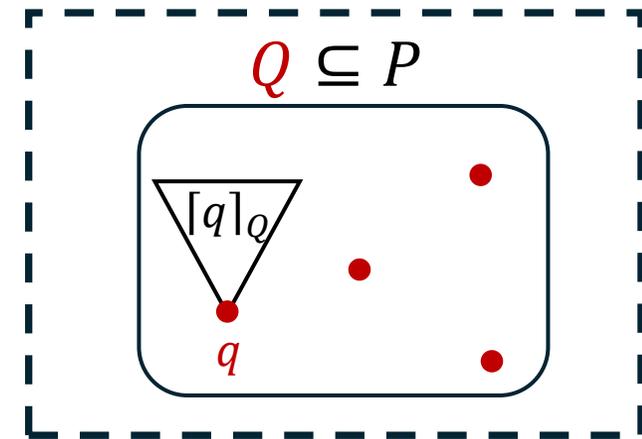
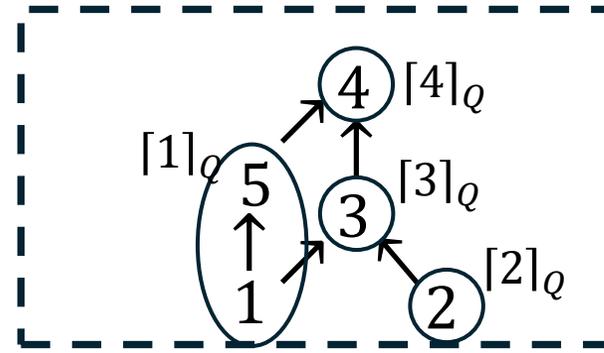
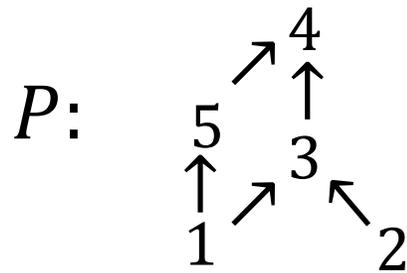
is the right adjoint of $\iota_Q: Q \hookrightarrow P$.

- $\lceil q \rceil_Q := \lfloor \cdot \rfloor_Q^{-1}(q) (= \{p \in P \mid \lfloor p \rfloor_Q = q\})$.

Example



Interior system
 \subseteq



Contraction functor

Definition

Let $Q \subseteq P$ be an interior system. The left Kan extension along $\lfloor \rfloor_Q$

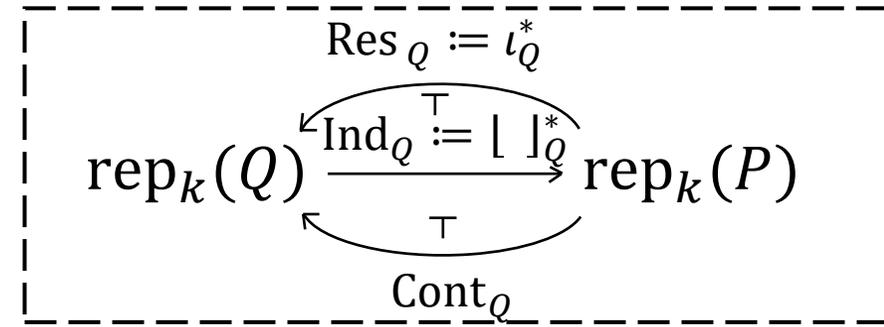
$$\text{Cont}_Q := \text{Lan}_{\lfloor \rfloor_Q} : \text{rep}_k(P) \rightarrow \text{rep}_k(Q)$$

is called ***contraction functor***.

$$\begin{array}{ccc}
 Q \begin{array}{c} \xrightarrow{\iota_Q} \\ \perp \\ \xleftarrow{\lfloor \rfloor_Q} \end{array} P & \rightsquigarrow & \text{rep}_k(Q) \begin{array}{c} \xleftarrow{\text{Ind}_Q := \lfloor \rfloor_Q^*} \\ \text{Res}_Q := \iota_Q^* \\ \xrightarrow{\text{Cont}_Q} \end{array} \text{rep}_k(P)
 \end{array}$$

- We have $\text{Cont}_Q(V)_q = \text{colim } V|_{\lfloor q \downarrow \rfloor_Q} \quad (\forall q \in Q)$.
- $\text{Ind}_Q := \lfloor \rfloor_Q^* : \text{rep}_k(P) \rightarrow \text{rep}_k(Q), V \mapsto V \circ \lfloor \rfloor_Q$. (This is the left Kan extension along ι_Q .)
- $\text{Res}_Q := \iota_Q^* : \text{rep}_k(P) \rightarrow \text{rep}_k(Q), V \mapsto V \circ \iota_Q$.

Basic properties of induced functors

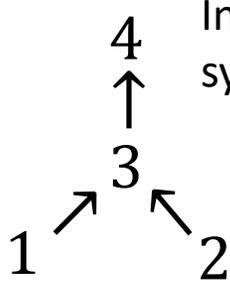
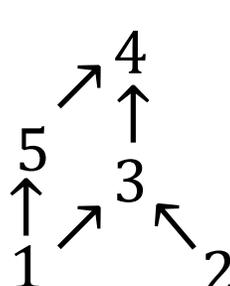
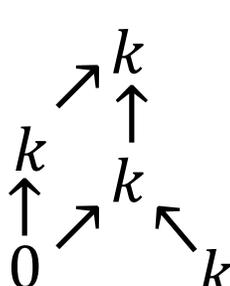


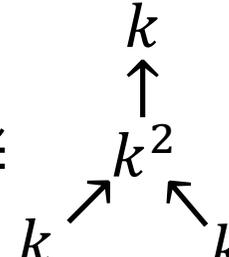
For an aligned interior system $Q \subseteq P$, the following hold.

Property	Ind_Q	Cont_Q
Exact	○	×
Right exact	○	○
Preserves indec. projectives	○	○
Preserves interval decomposability	○	×

Contraction functors do not preserve interval decomposability.

Example

For Q :  \subseteq Interior system $\subseteq P$:  and $V :=$  $\in \text{rep}_k(P)$,

$\text{Cont}_Q(V) \cong$  $\in \text{rep}_k(Q)$ is not interval-decomposable.

$$(\text{Cont}_Q(V))_3 := \text{colim} \left(V|_{\{p \in P \mid |p|_Q \leq 3\}} \right) = \text{colim} \left(\begin{array}{c} k \\ \uparrow \\ 0 \end{array} \begin{array}{c} \nearrow \\ \nearrow \\ \nearrow \end{array} \begin{array}{c} k \\ k \\ k \end{array} \leftarrow \begin{array}{c} k \\ k \\ k \end{array} \right) \cong k^2$$

\rightsquigarrow We introduce aligned interior systems

Aligned interior system

Definition

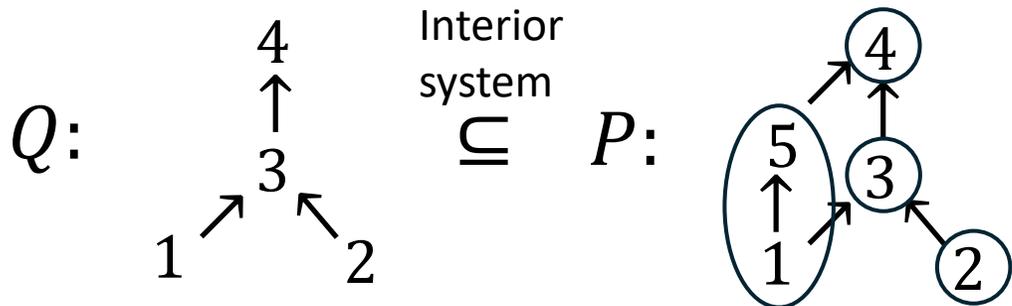
Let Q be an interior system of P . We say that Q is an **aligned interior system** if the following conditions are satisfied:

(AL1) $\forall q \in Q, [q]_Q (= \{p \in P \mid [p]_Q = q\})$ is filtered and

(AL2) $\forall q \in Q, [q^\downarrow]_Q = ([q]_Q)^\downarrow$ (\cong holds in general).

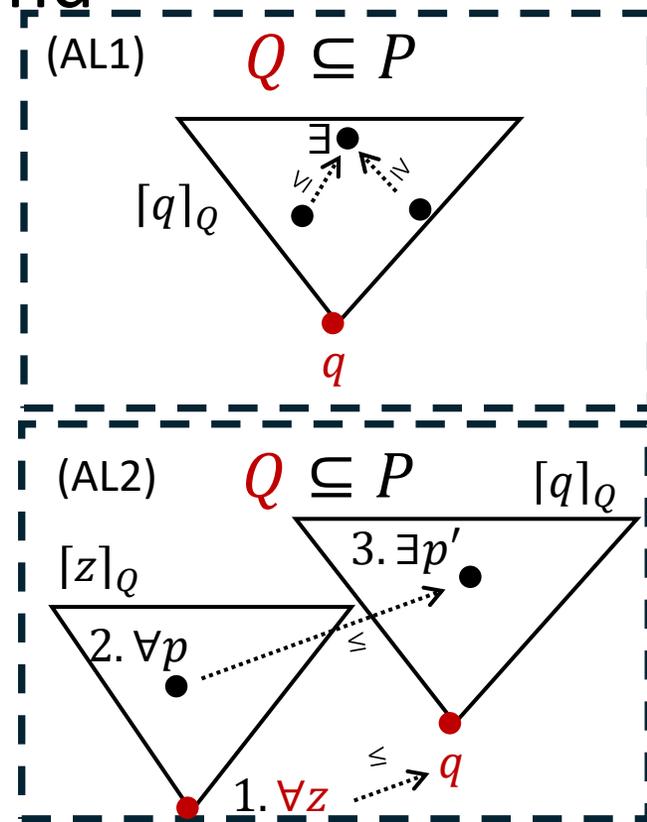
Example

(2) Q is **not** an aligned interior system of P .



$\cdot [3]_Q =$
 is not filtered.

\because Both (AL1) and (AL2) are not satisfied.



Aligned interior system

Remark

Let Q be an aligned interior system of P . Then Cont_Q is exact. This follows from (AL1) (the filtered condition).

Remark

Let $P = T_1 \times \cdots \times T_n$ be a product of totally ordered sets, and let $T'_i \subseteq T_i$ be a finite subset. We call $Q = T'_1 \times \cdots \times T'_n$ a finite aligned subgrid [BBH, Def 7.1]. Then $Q \subseteq Q^\uparrow$ is an aligned interior system.

Moreover, the contraction functor $\text{rep}_k Q^\uparrow \rightarrow \text{rep}_k Q$ defined in [BBH, Def 7.4] is a special case of Cont_Q .

Aligned interior system

Proposition[Aoki-T.]

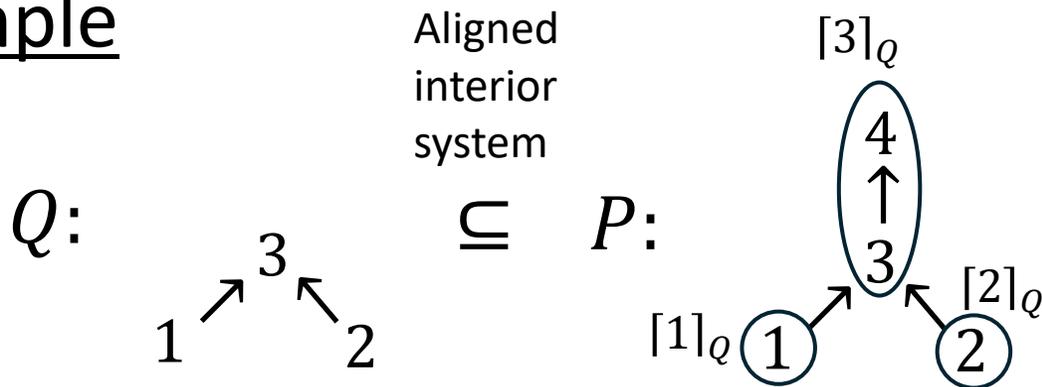
Let Q be an aligned interior system of P . Then, Cont_Q sends interval module to interval-decomposable modules. Indeed, for any interval I of P , we have

$$\text{Cont}_Q(k_I) \cong k_{\bar{I}^Q},$$

where $\bar{I}^Q \subseteq Q$ is a convex set given by

$$\bar{I}^Q := \{q \in Q \mid I \cap [q]_Q \neq \emptyset \text{ and } (I \cap [q]_Q)^\uparrow = I \cap [q]_Q \text{ in } [q]_Q\}.$$

Example



Let $I = Q$. Then $\bar{I}^Q = \{1,2\}$ (convex set).

$$\because (I \cap [1]_Q)^\uparrow = \{1\} = I \cap [1]_Q \text{ in } [1]_Q$$

$$(I \cap [2]_Q)^\uparrow = \{2\} = I \cap [2]_Q \text{ in } [2]_Q$$

$$(I \cap [3]_Q)^\uparrow = \{3,4\} \neq \{3\} = I \cap [3]_Q \text{ in } [3]_Q$$

Aligned interior system

Theorem[AT]

Let Q be an aligned interior system of P . Then, Ind_Q sends an interval resolutions to an interval resolution.

Corollary: Let P be a finite poset. If for any interval I in P , there exists an aligned interior system $Q_I \subseteq P$ s.t. Γk_I is an image of Ind_{Q_I} , then we have

$$\max\{\text{int-res-gldim}(Q_I) \mid I \text{ is interval of } P\} = \text{int-res-gldim}(P).$$

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(4) **Applications**

- Operations that preserve interval resolution global dimension of posets

Applications: Operations that preserve int-gl. dim. of posets

Theorem[AT]

Let $n \geq 3$.

(1) The interval resolution global dimensions of the following posets are the same:

$$P': \begin{array}{c} \nearrow \\ \vdots \\ \rightarrow \end{array} a_1 \rightarrow a_2 \rightarrow a_3 \begin{array}{c} \nearrow \\ \vdots \\ \searrow \end{array} \subseteq P: \begin{array}{c} \nearrow \\ \vdots \\ \rightarrow \end{array} a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_n \begin{array}{c} \nearrow \\ \vdots \\ \searrow \end{array}$$

(2) The interval resolution global dimensions of the following posets are the same:

$$P': \begin{array}{c} \nearrow \\ \vdots \\ \rightarrow \end{array} a_1 \Leftrightarrow a_2 \Leftrightarrow a_3 \subseteq P: \begin{array}{c} \nearrow \\ \vdots \\ \rightarrow \end{array} a_1 \Leftrightarrow a_2 \Leftrightarrow \dots \Leftrightarrow a_n,$$

where \Leftrightarrow is either \rightarrow or \leftarrow .

Applications: Operations that preserve int-gl. dim. of posets

Example: Let P_n be a poset with the Hasse diagram:

$$P_n := \begin{array}{ccccccc} & & & \cdot & & & \\ & & & | & & & \\ \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot \\ & & \downarrow & & \uparrow & & & & \\ & & a_1 & \rightarrow & \cdots & \rightarrow & a_n & & \end{array}$$

Then, we have $\text{int-res-gldim}(P_n) = 1$. Indeed, we have
 $\text{int-res-gldim}(P_n) = \text{int-res-gldim}(P_3) = 1$.

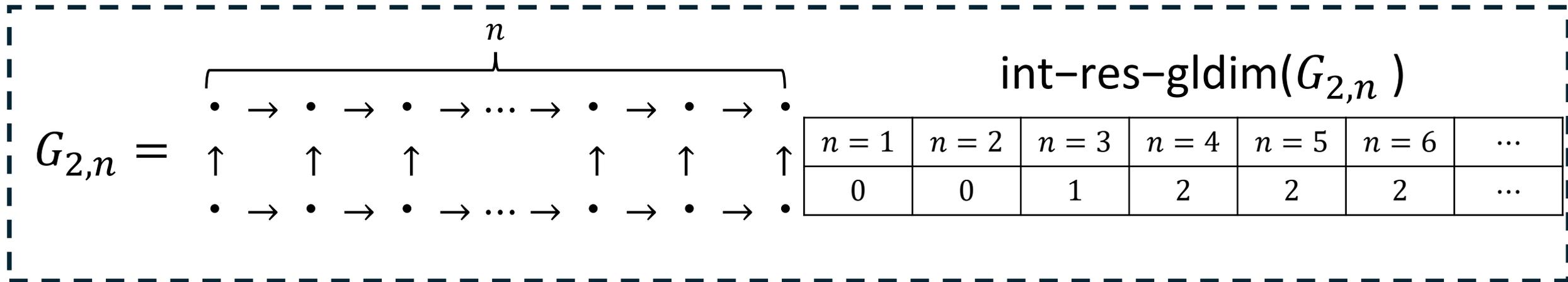
$$P_3 = \begin{array}{ccccccc} & & & \cdot & & & \\ & & & | & & & \\ \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot & \rightarrow & \cdot \\ & & \downarrow & & \uparrow & & & & \\ & & a_1 & \rightarrow & a_2 & \rightarrow & a_3 & & \end{array}$$

Applications: Operations that preserve int-gl. dim. of posets

Remark: Using contraction functors for grid posets, [BDHS '25] show that the interval resolution global dimension of

$$G_{2,n} := \{1 < 2\} \times \{1 < \dots, < n\}$$

stabilizes as n increases, as conjectured in [AENY '23, Conj. 4.11].

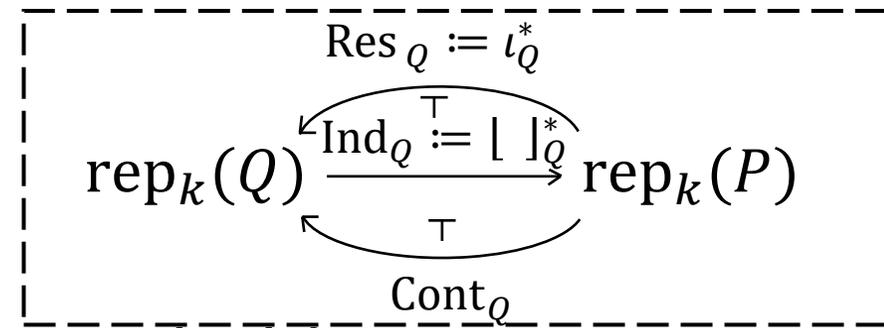


Blanchette, B., Desrochers, J., Hanson, E. J., & Scoccola, L. (2025). Stabilization of the Spread-Global Dimension. *arXiv preprint arXiv:2506.01828*.

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Summary

- We introduce **aligned** interior systems.



For an aligned interior system $Q \subseteq P$, the following hold.

Property	Ind_Q	Cont_Q
Exact	○	○
Preserves indec. projectives	○	○
Preserves interval-decomposability	○	○

- In addition, Ind_Q preserves interval resolutions.
- We give operations of posets that preserves interval resolution global dimensions.

Thank you for listening.