Proposal of Bipath Persistent Homology: Visualization, Algorithm, and Stability

2025/10/21

Tohoku University
Shunsuke Tada

- T. Aoki, E. G. Escolar, and S. Tada. "Bipath persistence" Japan J. Indust. Appl. Math. 42, 453–486 (2025).
- S. Tada. Stability of Bipath Persistence Diagrams. arXiv: 2503.01614, 2025.

Research

・Studying multiparameter persistent homology using representation theory of associative algebras.

・ Recently also interested in causal inference.

・Since April 2025, studying the stability of causal graphs under Kano-sensei.

Additional notes

I have experience of working at a mountain lodge. (Kano-san also has similar experience.)

2014.4--2014.11

An aim is talking about

Bipath persistent homology

which is an extension of persistent homology(PH).

- A Visualization (bipath persistence diagrams (PDs))
- Computation of bipath persistence PDs

(Joint work with Toshitaka Aoki, Emerson G. Escolar)

- Stability of bipath PDs

<u>Contents</u>

(1)Introduction: TDA, PH and bipath PH.

(2)Bipath PDs/computation/stability properties.

(3)Summary.

Introduction (TDA)

Topological Data Analysis (TDA) is a field of data analysis that utilizes concepts of topology.

Topology:	Top. Sp.	$\longrightarrow$	Features (e.g., homology)
TDA:	Data	$\longrightarrow$	Topological features

Example.

・Persistent homology
・Mapper
・Topological flow data analysis

Introduction (TDA)

Topological Data Analysis (TDA) is a field of data analysis that utilizes concepts of topology.

Topology: Top. Sp. $\longrightarrow$ Features (e.g., homology)

TDA: Data $\longrightarrow$ Topological features

Example.

・Persistent homology

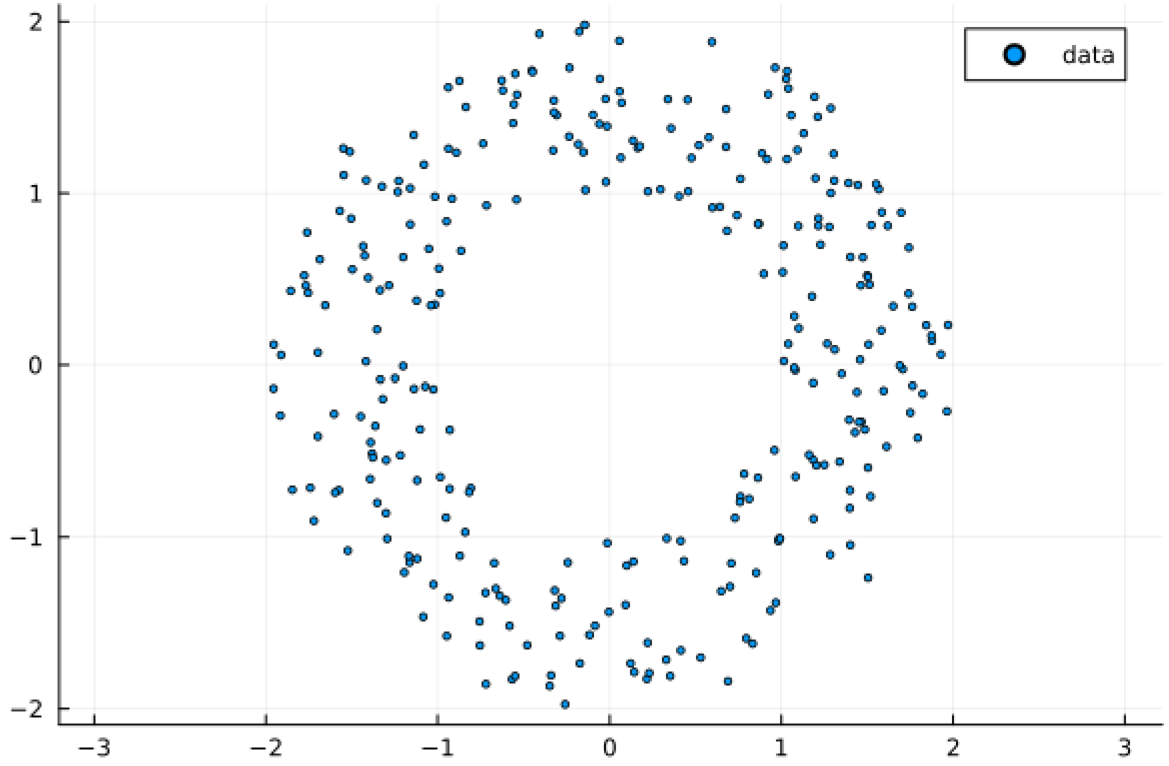
Data $\longrightarrow$ Persistence Diagram

Ex. Point cloud,
Image data

Introduction (PH)

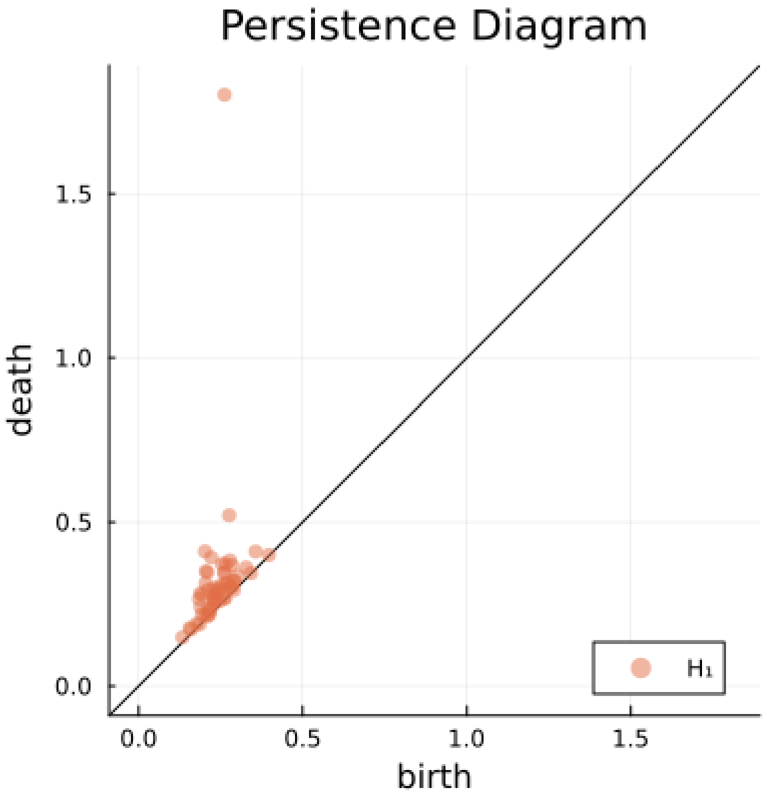
・Persistent homology captures the “shape” (connected components, holes or voids) of data by a *persistence diagram* (PD).

・Inverse analysis, stability property



Large hole in data ↔ Point far from the diagonal

Small hole in data ↔ Point close to the diagonal



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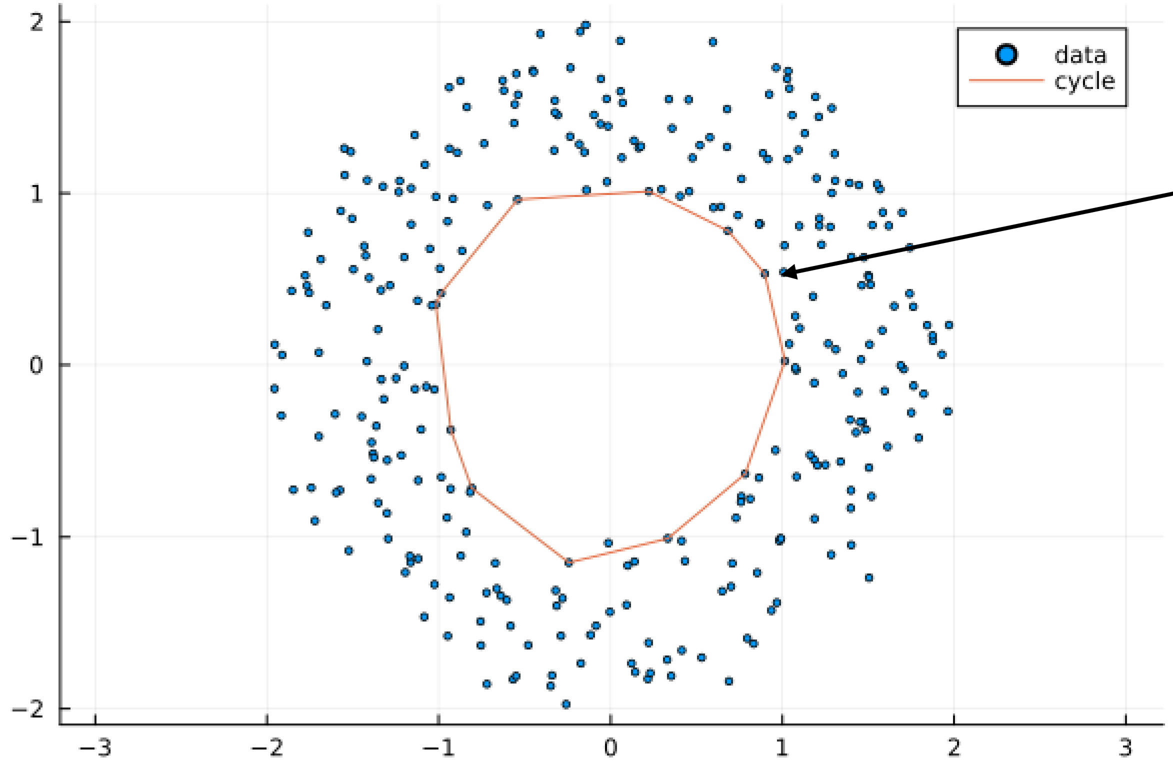

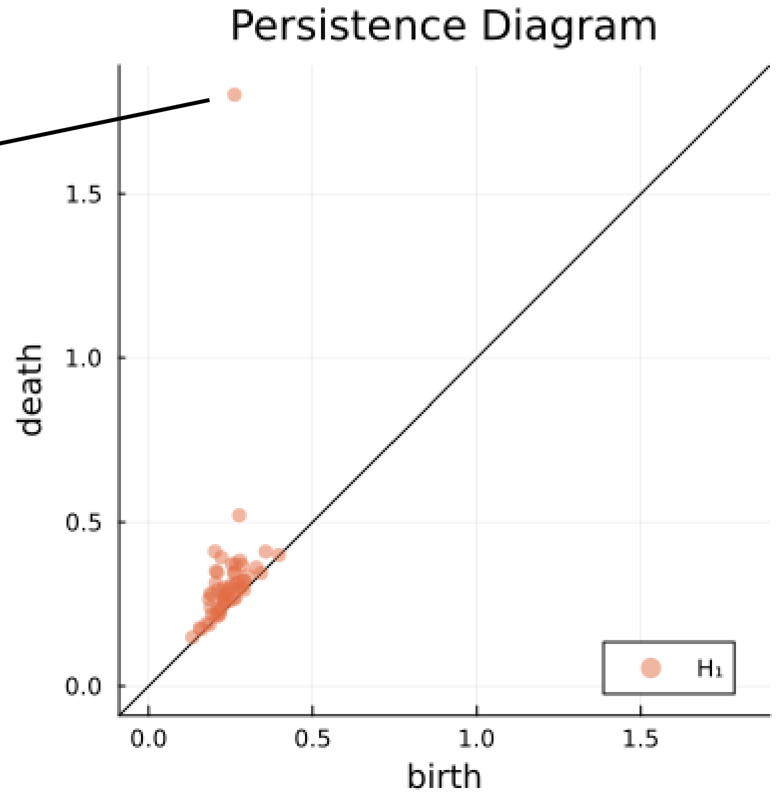

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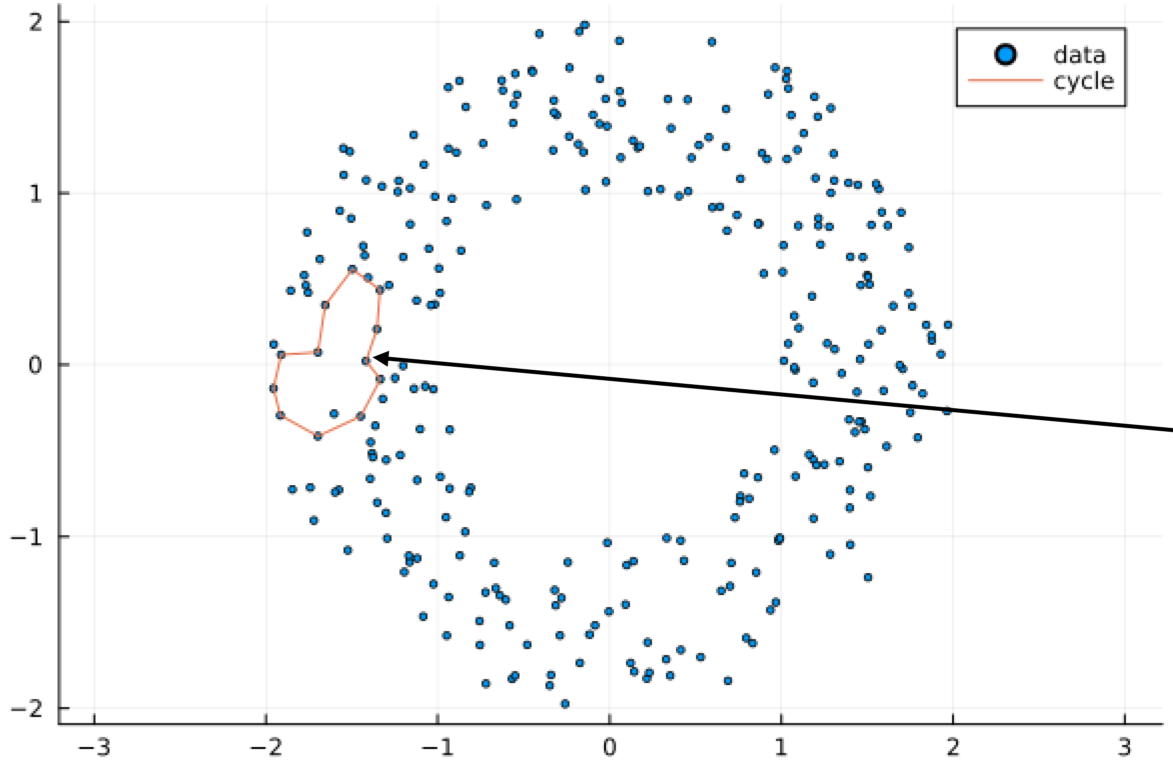

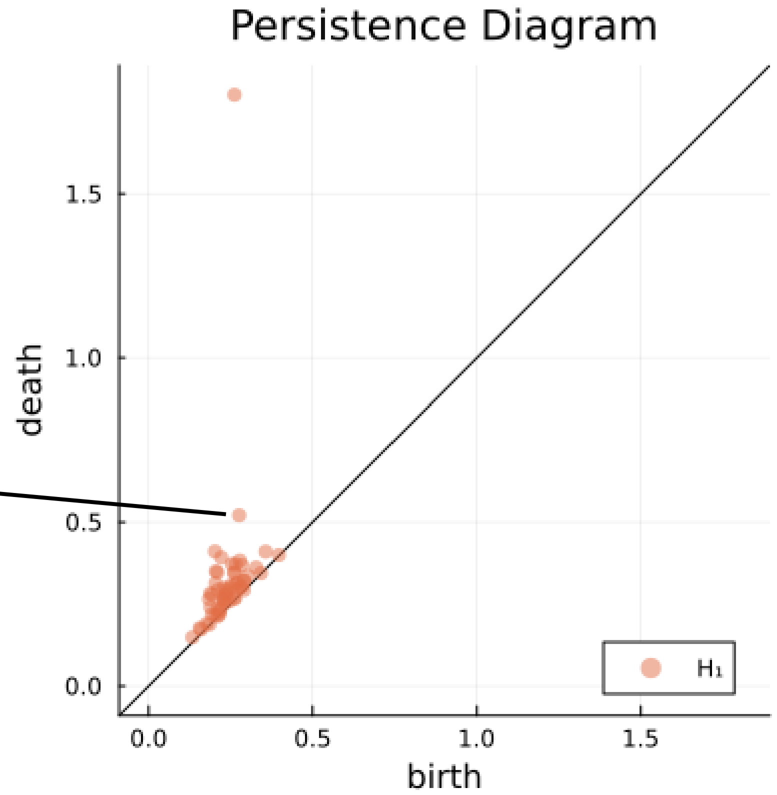

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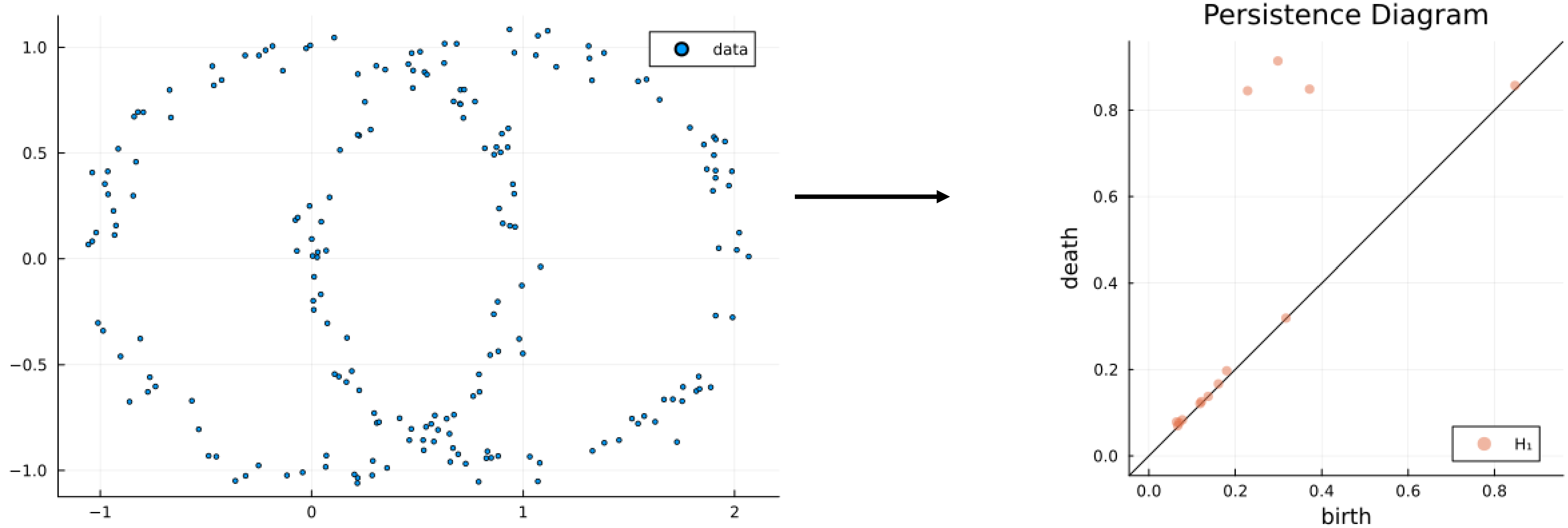

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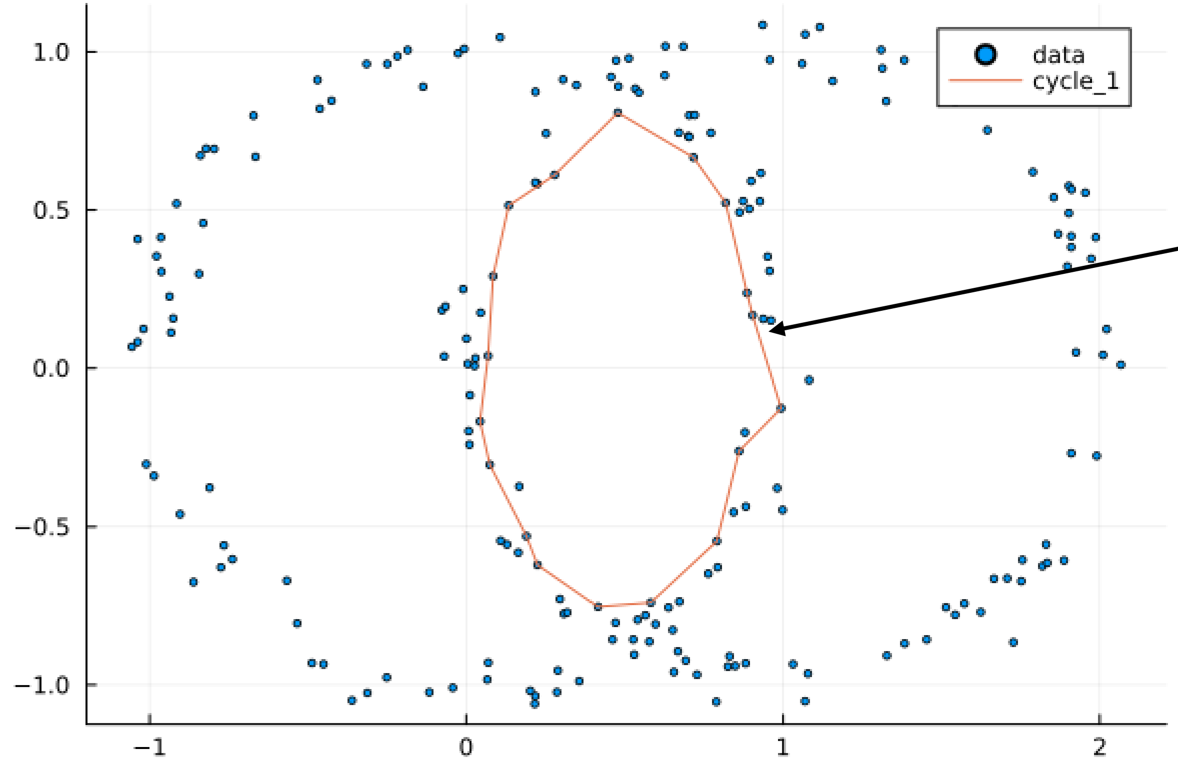

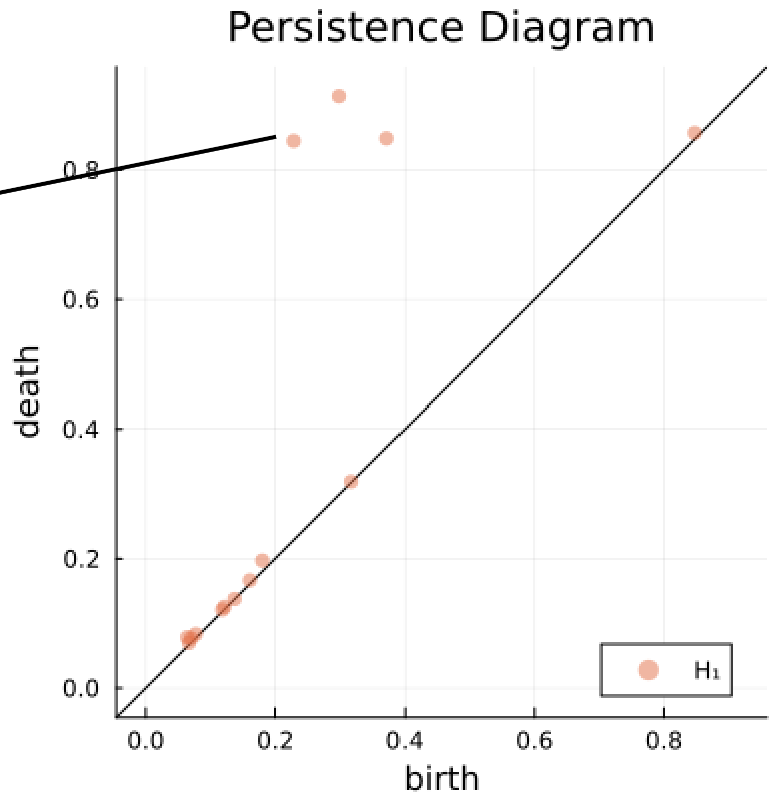

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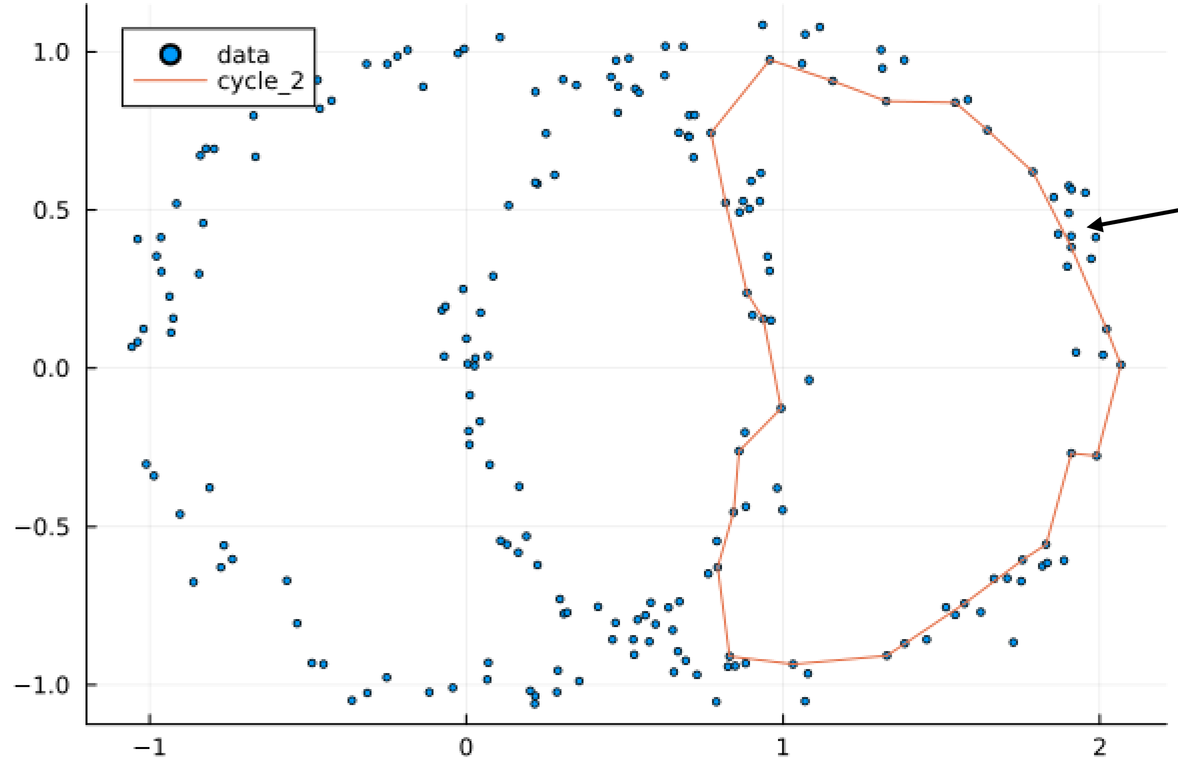

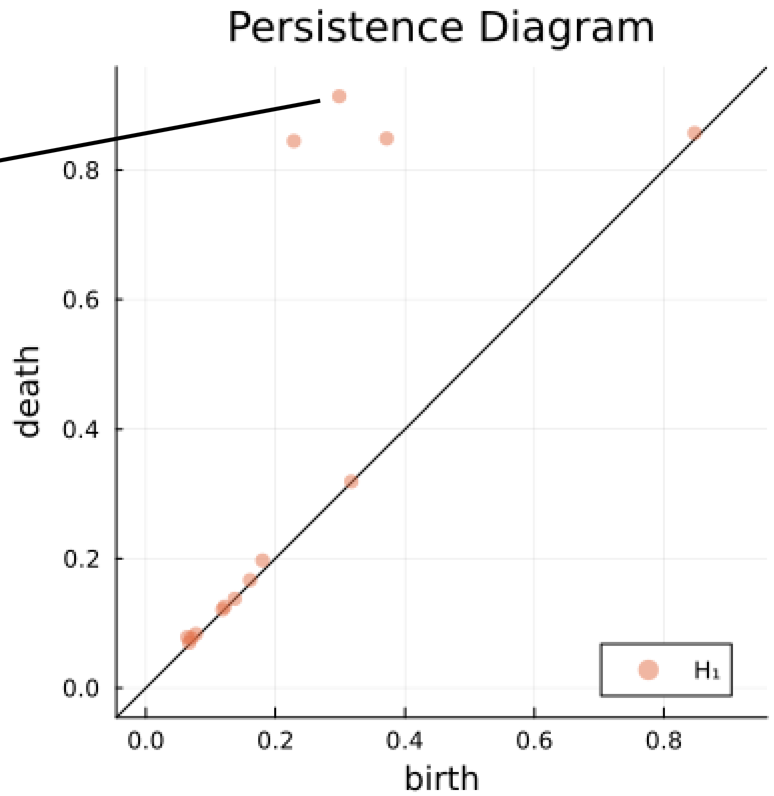

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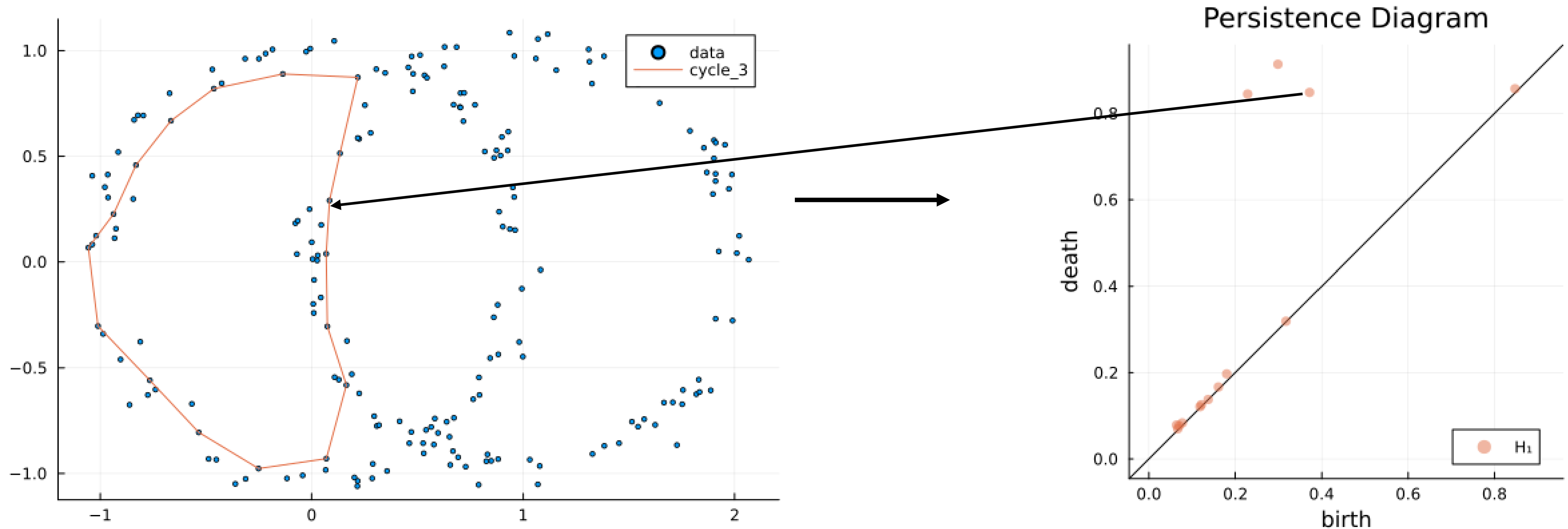

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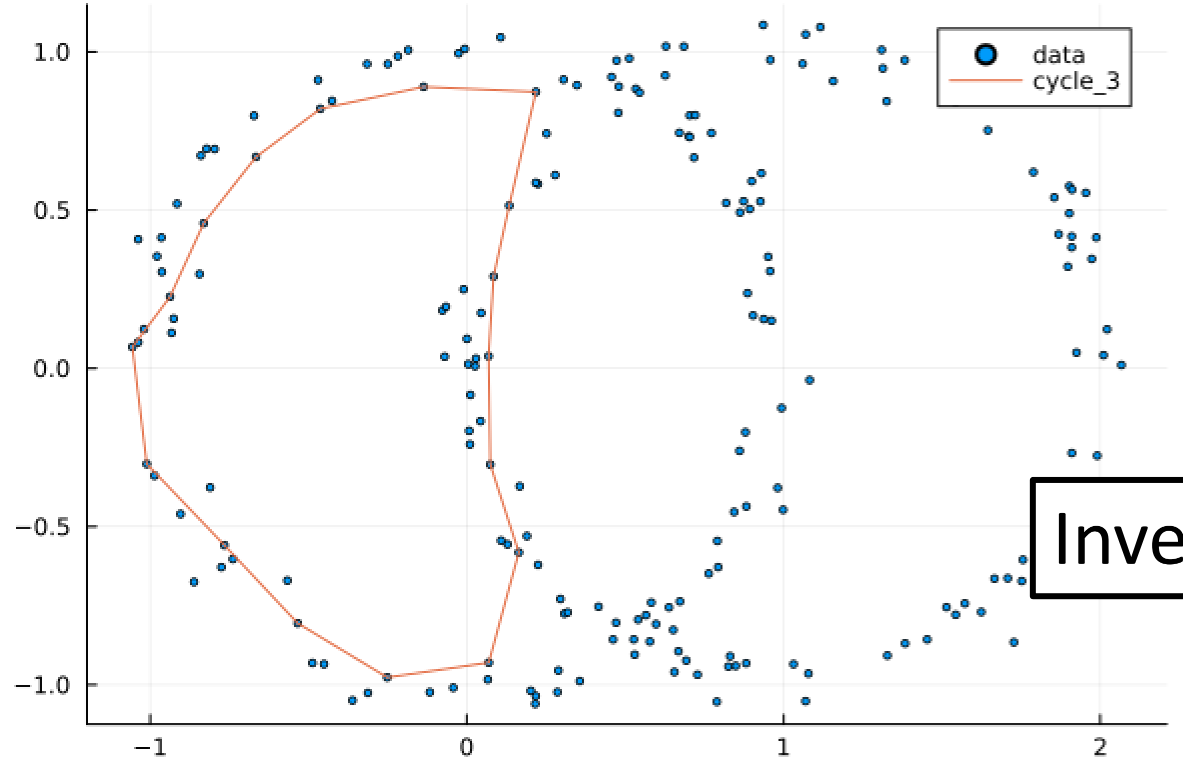
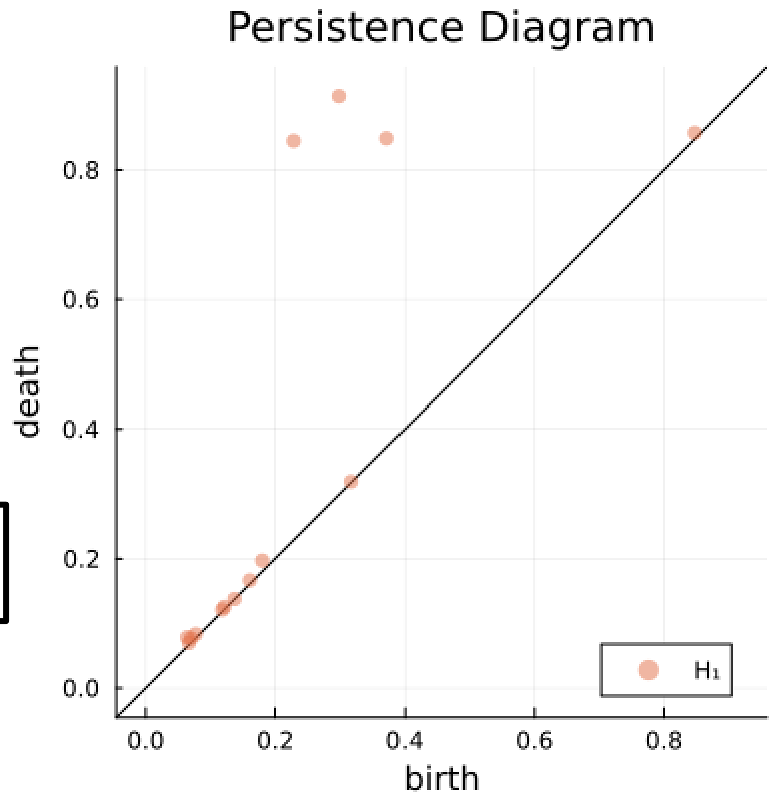
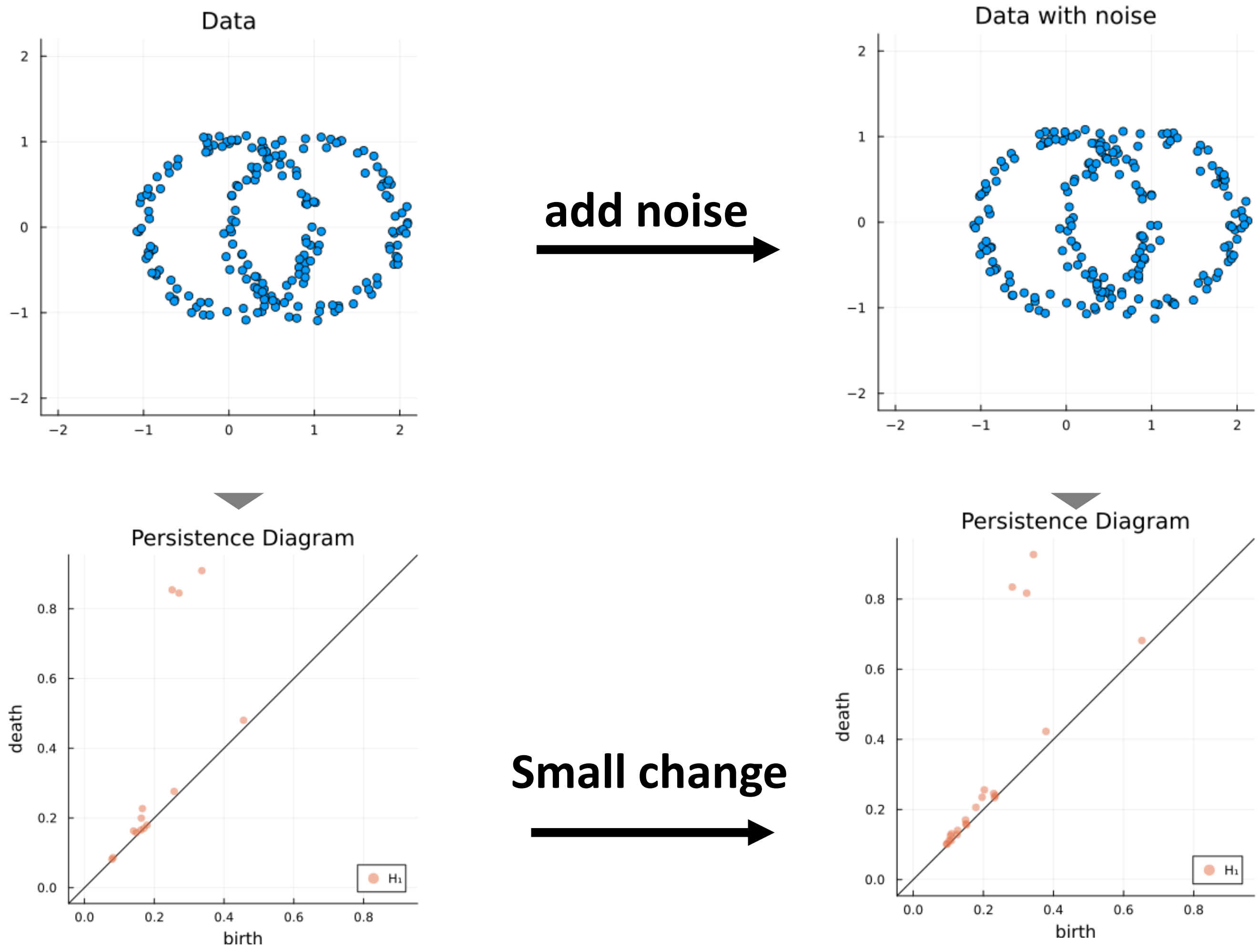


Inverse analysis



Introduction (PH)

stability property justifies the use of PH to noisy data.



Introduction (PH)

・ Material science

・ Evolutional biology

・ Computational gastronomy

and others…

・Yasuaki Hiraoka, Takenobu Nakamura, Akihiko Hirata, Emerson G. Escolar, Kaname Matsue, and Yasumasa Nishiura. Hierarchical structures of amorphous solids characterized by persistent homology. *Proceedings of the National Academy of Sciences*, *113*(26), 7035-7040, 2016.

・Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. Topology of viral evolution. Proceedings of the National Academy of Sciences, 110(46):18566–18571, 2013

・Emerson G. Escolar, Yuta Shimada, and Masahiro Yuasa. A topological analysis of the space of recipes. International Journal of Gastronomy and Food Science, 39:101088, 2025.

Introduction (PH)

・ GUDHI (C++/Python)

・ HomCloud (Python)

・ Ripser (C++/Python)

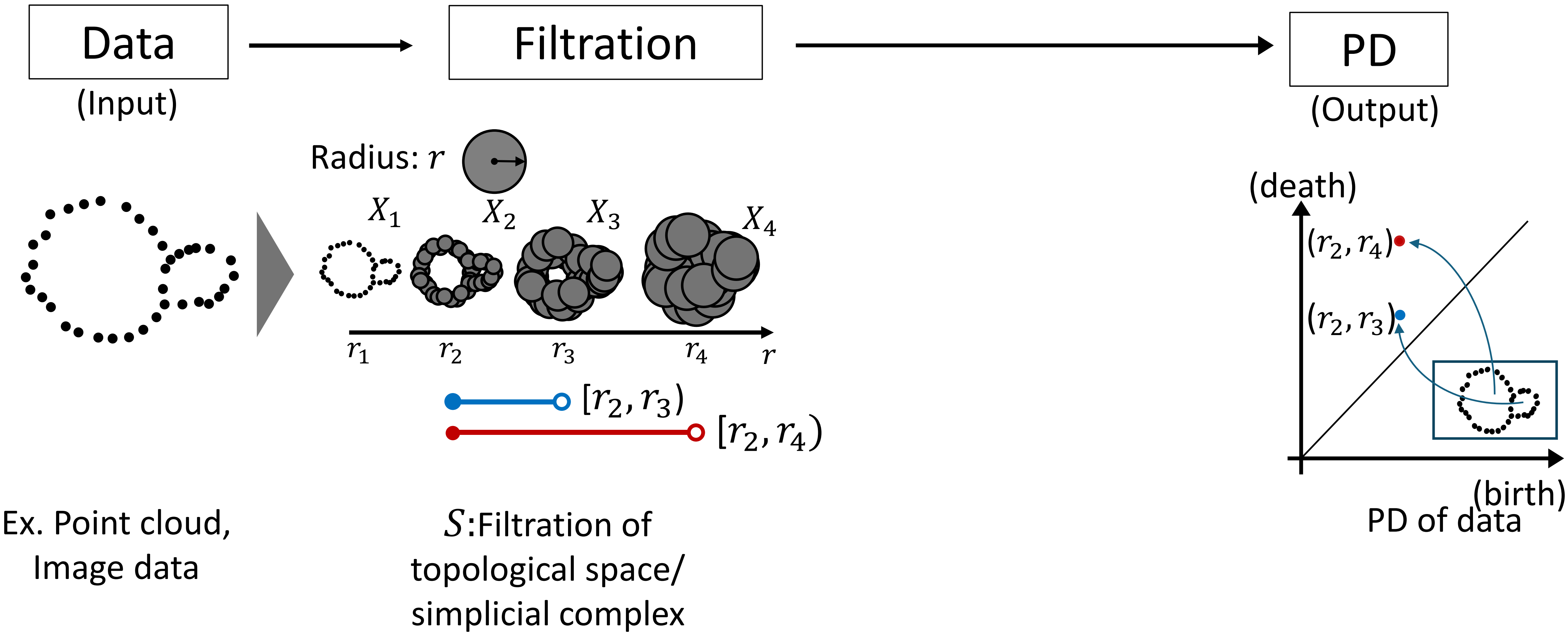
・ Ripserer.jl (Julia)

and others…

$\rightsquigarrow$ How to get PDs.

Introduction (PH)

PH: Persistent homology
PD: Persistence diagram



Introduction (PH)

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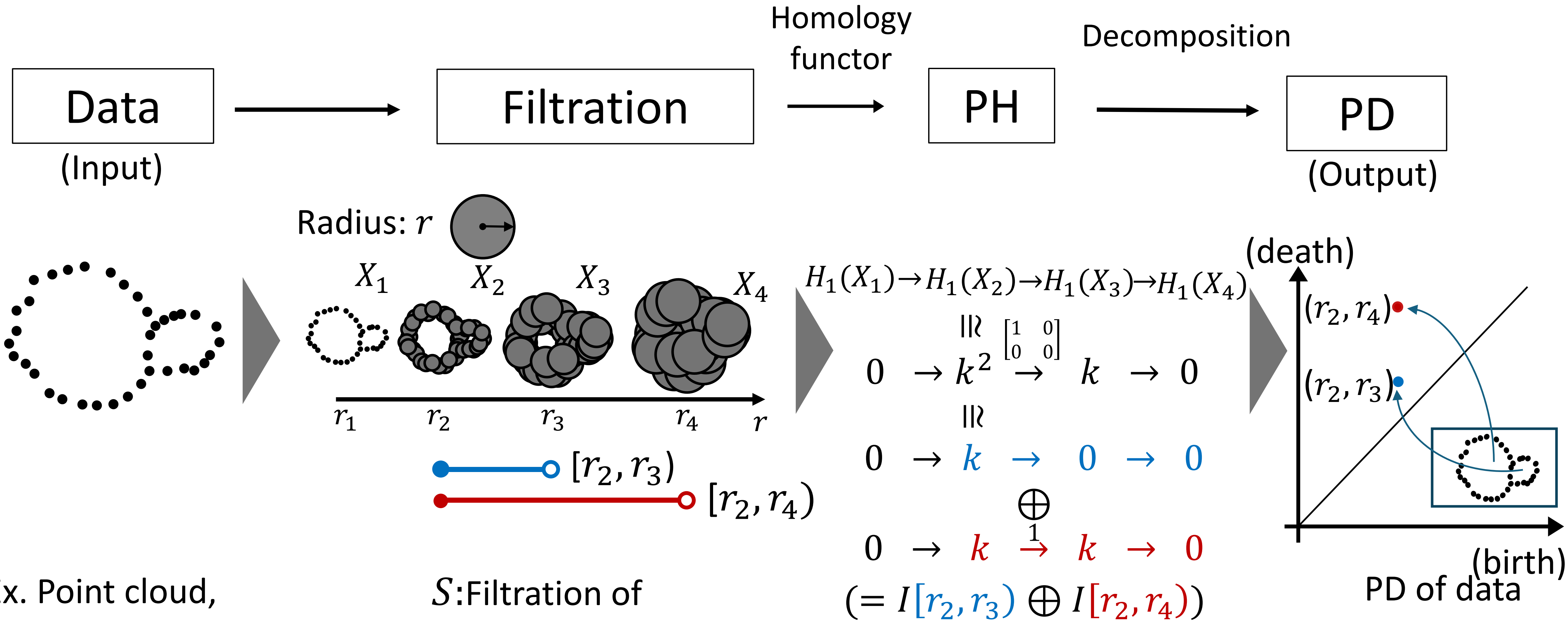


Ex. Point cloud,
Image data

S:Filtration of
topological space/
simplicial complex

Every persistent homology decomposes into intervals (cf. Gabriel's theorem).

Introduction (PH, summary)

$$\underset{\text{(Input)}}{\boxed{\text{Data}}} \longrightarrow \boxed{\text{Filtration}} \xrightarrow{\substack{\text{Homology}\\ \text{functor}}} \boxed{\text{PH}} \xrightarrow{\text{Decomposition}} \underset{\text{(Output)}}{\boxed{\text{PD}}}$$

・PH is a central tool in TDA, providing topological features in data by the PD (intervals).

・Inverse analysis corresponds points in PDs to topological features in a given data.

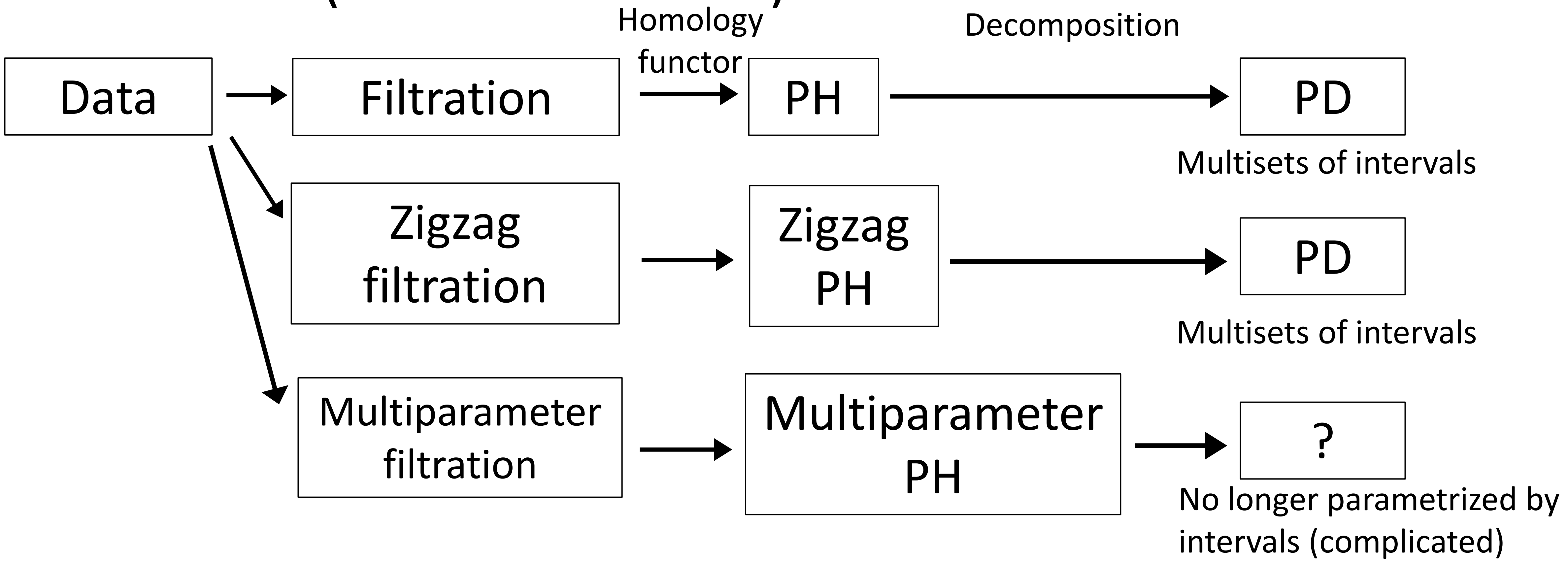
・The stability property justifies the use of PH to noisy data.

$\rightsquigarrow$ PH is a nice tool! A lot of applications.

$\rightsquigarrow$ Some extensions of PH.

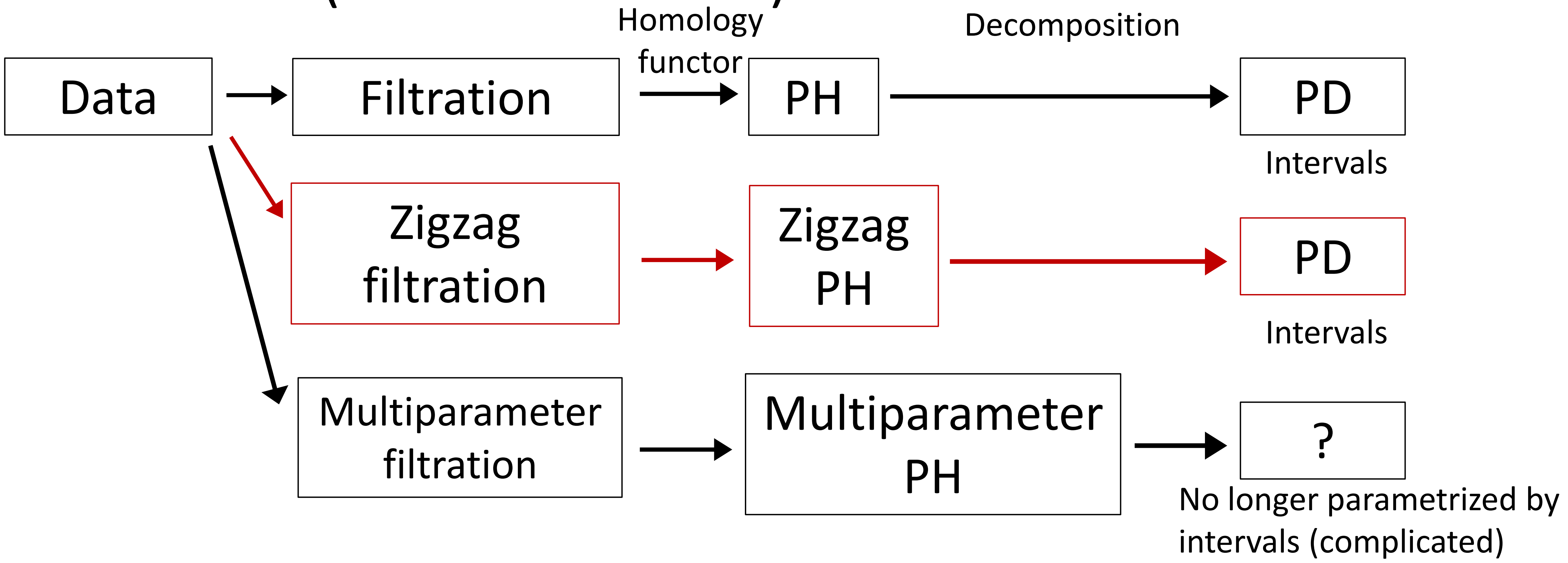
Introduction (Extensions of PH)

PH: Persistent homology
PD: Persistence diagram



Introduction (Extensions of PH)

PH: Persistent homology
PD: Persistence diagram



$$S: S_1 \supset S_2 \subset S_3 \supset S_4 \subset S_5$$

Zigzag filtration

e.g., Temporal network

$$H_i(S) = H_i(S_{r_1}) \leftarrow H_i(S_{r_2}) \rightarrow H_i(S_{r_3}) \leftarrow H_i(S_{r_4}) \rightarrow H_i(S_{r_5})$$

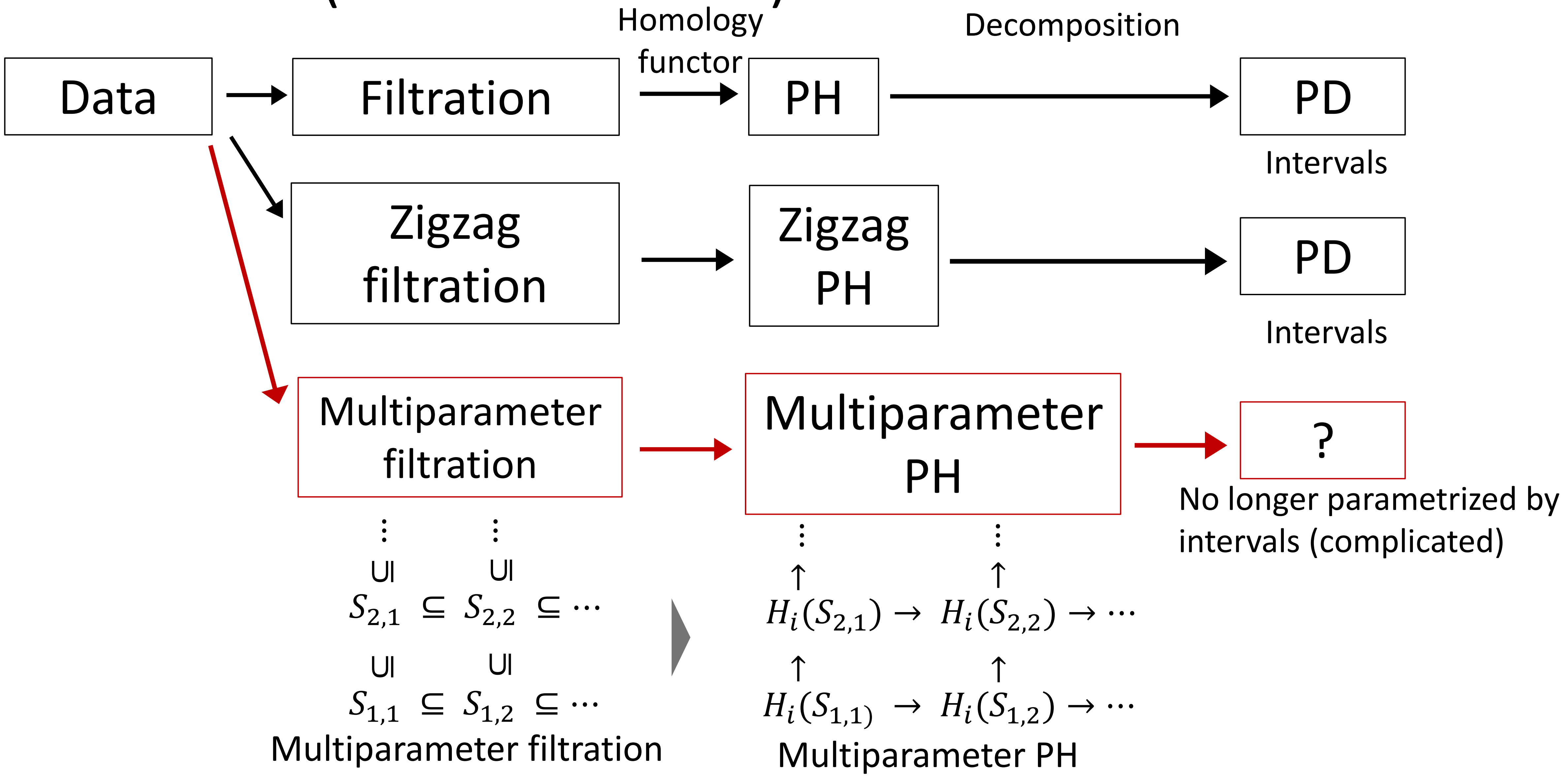
$$\cong \bigoplus I[b_i, d_i]^{m_{b_i, d_i}}$$ (Interval-decomposable)

Zigzag PH

Myers, A., Muñoz, D., Khasawneh, F. A., & Munch, E. (2023). Temporal network analysis using zigzag persistence. *EPJ Data Science*, *12*(1), 6.
Gunnar Carlsson, and Vin De Silva. Zigzag persistence. Foundations of computational mathematics 10 (2010): 367-405.

PH: Persistent homology
PD: Persistence diagram

Introduction (Extensions of PH)



Gunnar Carlsson, and Afra Zomorodian. The Theory of Multidimensional Persistence. *Discrete Comput Geom* **42**, 71–93 (2009).

Introduction (Extensions of PH)

An indecomposable module:

This is not interval.

$$\begin{tikzcd}
K \arrow[r] & K \arrow[r] & K & & & & & & & & \\
K \arrow[r] \arrow[u] & K \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[u] & K^2 \arrow[r, "{[1\ 0]}"] \arrow[u, "{[1\ 1]}"] & K \arrow[r] & K & & & & & & \\
K \arrow[r] \arrow[u] & K \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[u] & K^2 \arrow[r] \arrow[u] & K^2 \arrow[r] \arrow[u, "{[1\ 0]}"] & K^2 \arrow[r, "{[0\ 1]}"] \arrow[u, "{[1\ 0]}"] & K \arrow[r] & K & & & & \\
& & & & K \arrow[r] \arrow[u, "{\begin{bmatrix}1\\0\end{bmatrix}}"] & {\color{red}0} \arrow[r] \arrow[u] & K \arrow[u] & & & & \\
& & & & K \arrow[r] \arrow[u] & K \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[u] & K^2 \arrow[r] \arrow[u, "{[0\ 1]}"] & K^2 \arrow[r] & K^2 \arrow[r, "{[0\ 1]}"] & K \arrow[r] & K \\
& & & & & & K \arrow[r] \arrow[u, "{\begin{bmatrix}0\\1\end{bmatrix}}"] & K \arrow[r, "{\begin{bmatrix}0\\1\end{bmatrix}}"] \arrow[u, "{\begin{bmatrix}0\\1\end{bmatrix}}"] & K^2 \arrow[r, "{[0\ 1]}"] \arrow[u] & K \arrow[r] \arrow[u] & K \arrow[u] \\
& & & & & & & & K \arrow[r] \arrow[u, "{\begin{bmatrix}1\\1\end{bmatrix}}"] & K \arrow[r] \arrow[u] & K \arrow[u]
\end{tikzcd}$$

Mickaël Buchet, and Emerson G. Escolar. Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module. *Journal of Applied and Computational Topology.*

Introduction (Extensions of PH)

It is complicated to classify all the indecomposable module.
(wild representation type)

$$\begin{array}{cccccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
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\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
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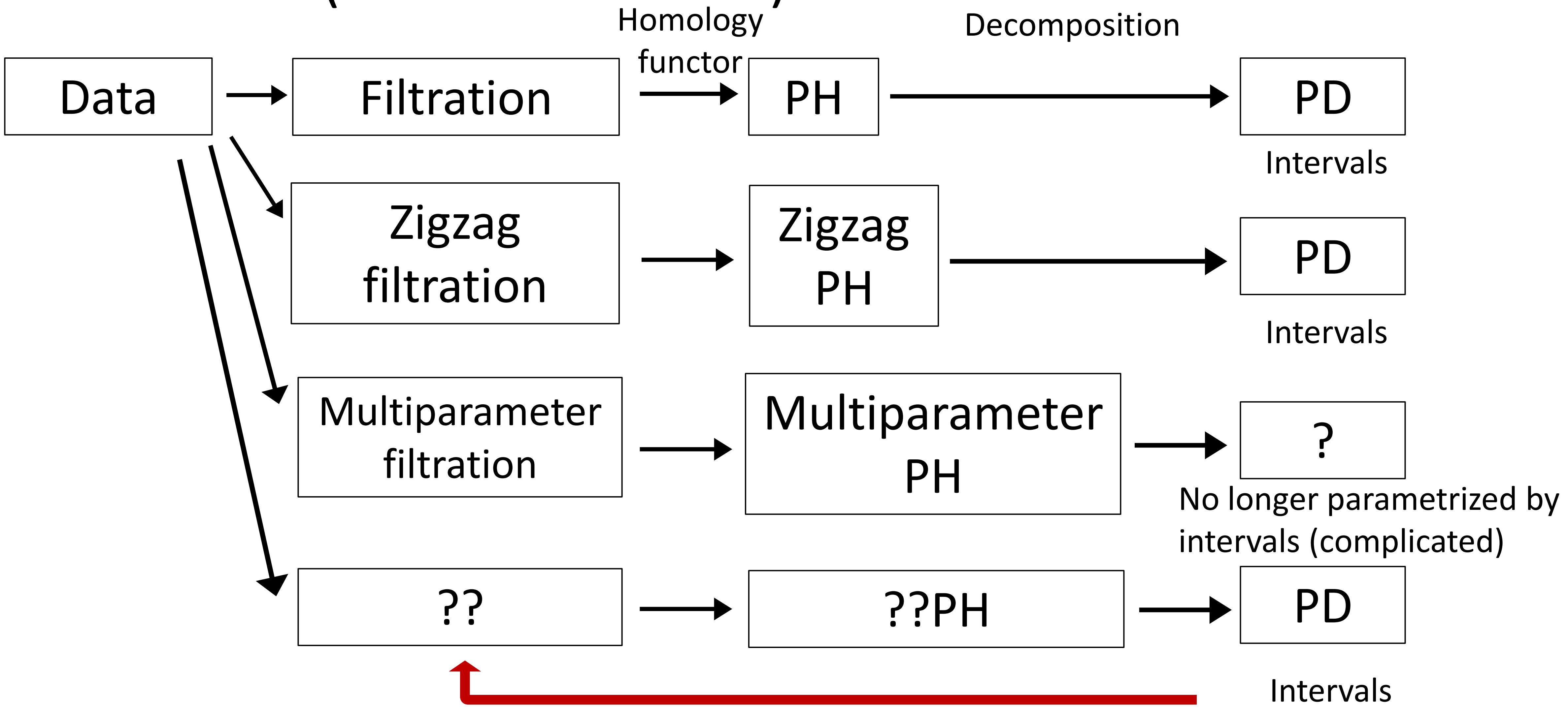
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow
\end{array}$$

・Zbigniew Leszczyński. On the representation type of tensor product algebras. Fundamenta Mathematicae, 144(2):143–161, 1994.
・Zbigniew Leszczyński and Andrzej Skowroński. Tame triangular matrix algebras. Colloquium Mathematicum, 86(2):259 303, 2000.
・Ulrich Bauer, Magnus B. Botnan, Steffen Oppermann, and Johan Steen. Cotorsion torsion triples and the representation theory of filtered hierarchical clustering. Advances in Mathematics, 369:107171, 2020.

PH: Persistent homology
PD: Persistence diagram

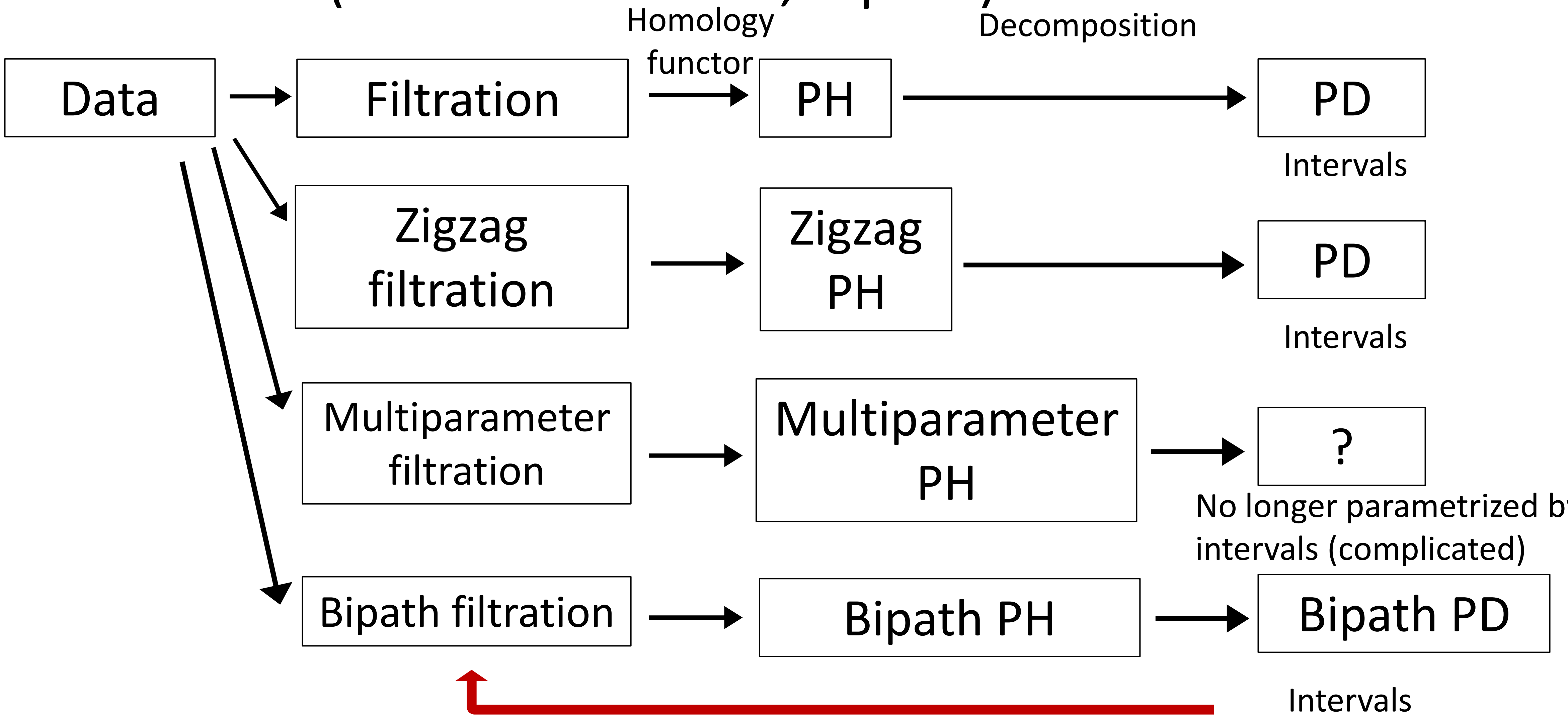
Introduction (Extensions of PH)



Do we have other arrangement of spaces like standard/zigzag filtration?

PH: Persistent homology
PD: Persistence diagram

Introduction (Extensions of PH, bipath)



<u>Bipath persistent homology</u> as a new framework [Aoki-Escolar-T, 25].

Introduction (Extensions of PH, bipath)

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[dr, hook] & \\
S: & S_{\hat{0}} \arrow[ur, hook] \arrow[dr, hook] & & & & & S_{\hat{1}} \\
 & & S_{1'} \arrow[r, hook] & S_{2'} \arrow[r, hook] & \cdots \arrow[r, hook] & S_{m'} \arrow[ur, hook] &
\end{tikzcd}
$$

Bipath filtration

||

$$
\begin{array}{ll}
 & S_{\hat{0}} \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{\hat{1}} \\
S: & \| \hspace{9cm} \| \\
 & S_{\hat{0}} \hookrightarrow S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} \hookrightarrow S_{\hat{1}}
\end{array}
$$

A pair of two filtrations sharing the same spaces at the ends.

Introduction (Extensions of PH, bipath)

We can consider a *bipath persistent homology* (bipath PH) of a filtration.

$$
\begin{tikzcd}[column sep=small]
 & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[dr, hook] & \\
S_{\hat{0}} \arrow[ur, hook] \arrow[dr, hook] & & & & & S_{\hat{1}} \\
 & S_{1'} \arrow[r, hook] & S_{2'} \arrow[r, hook] & \cdots \arrow[r, hook] & S_{m'} \arrow[ur, hook] &
\end{tikzcd}
$$

Homology functor

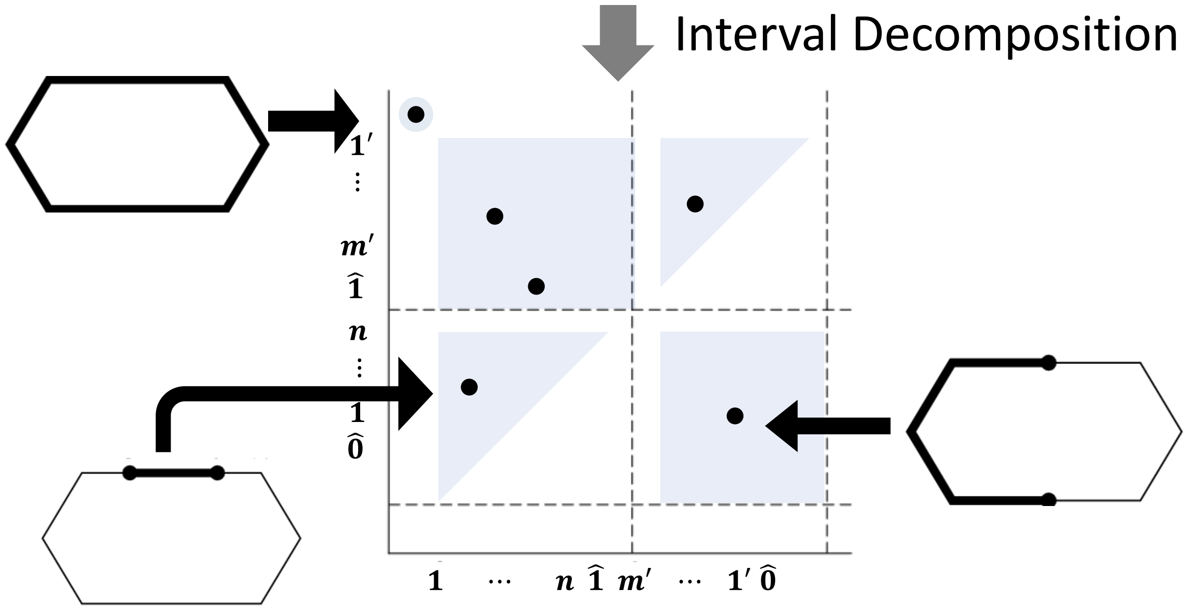
$$
\begin{tikzcd}[column sep=small]
 & H_q(S_1;k) \arrow[r] & H_q(S_2;k) \arrow[r] & \cdots \arrow[r] & H_q(S_n;k) \arrow[dr] & \\
H_q(S_{\hat{0}};k) \arrow[ur] \arrow[dr] & & & & & H_q(S_{\hat{1}};k) \\
 & H_q(S_{1'};k) \arrow[r] & H_q(S_{2'};k) \arrow[r] & \cdots \arrow[r] & H_q(S_{m'};k) \arrow[ur] &
\end{tikzcd}
$$

Bipath PH

Introduction (Extensions of PH, bipath)

We can get a *Bipath Persistence Diagram* (Bipath PD).

$$\begin{tikzcd}[column sep=small]
 & H_q(S_1;k) \arrow[r] & H_q(S_2;k) \arrow[r] & \cdots \arrow[r] & H_q(S_n;k) \arrow[dr] & \\
H_q(S_{\hat{0}};k) \arrow[ur] \arrow[dr] & & & & & H_q(S_{\hat{1}};k) \\
 & H_q(S_{1'};k) \arrow[r] & H_q(S_{2'};k) \arrow[r] & \cdots \arrow[r] & H_q(S_{m'};k) \arrow[ur] &
\end{tikzcd}$$

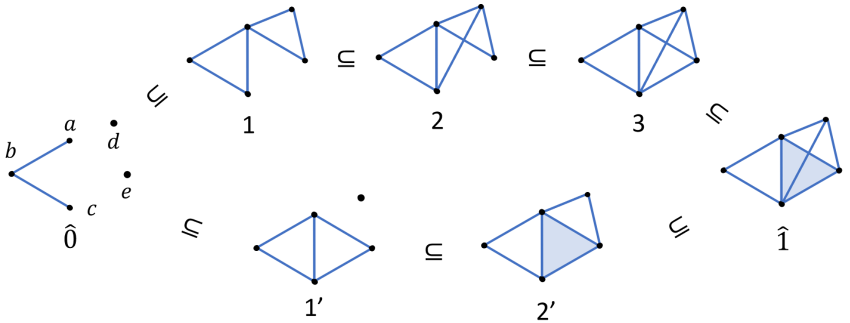


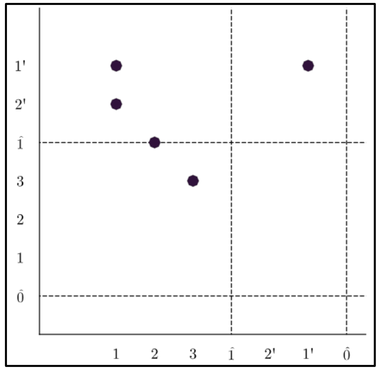
Bipath PD

Introduction (Extensions of PH, bipath)

Interval decomposability	○
Visualization (Bipath PD)	○
Algorithm (implementation)	○
Stability theorem for bipath PDs	○
Inverse analysis	-
Application	-

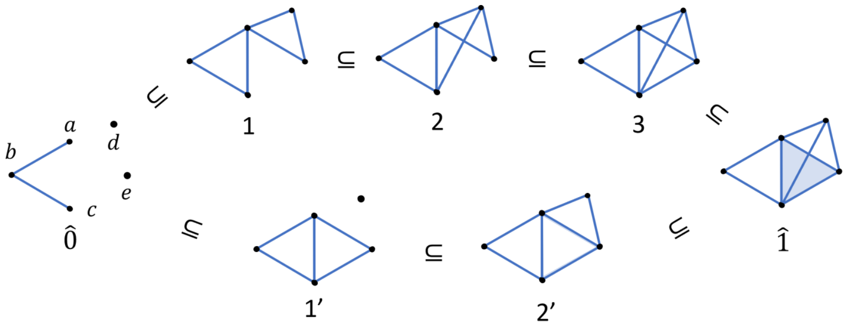
Implementation

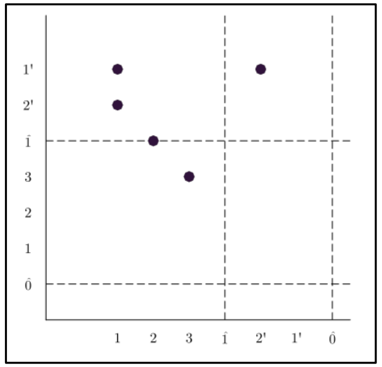




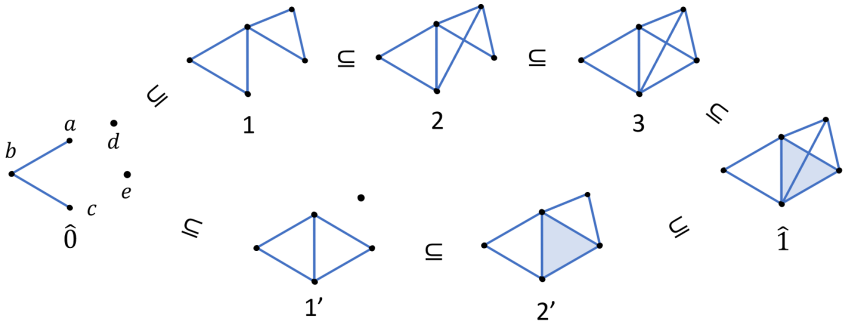
Introduction (Extensions of PH, bipath)

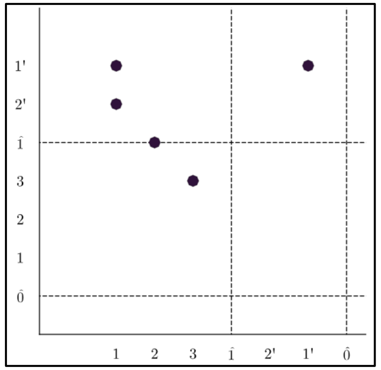
Stability





Implementation





Introduction (Extensions of PH, bipath)

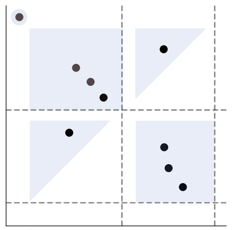
Interval decomposability	○
Visualization (Bipath PD)	○
Algorithm (implementation)	○
Stability theorem for bipath PDs	○
Inverse analysis	-
Application	-

Implementation

Data → Bi-filtration → Bipath filtration → Bipath PD

$$\begin{array}{ccccccc}
S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,3} & \to & S_{2,3} & \to & S_{3,3} & \to & S_{4,3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,2} & \to & S_{2,2} & \to & S_{3,2} & \to & S_{4,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}$$

$$\begin{array}{ccccccc}
S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
\uparrow & & & & & & \uparrow \\
S_{1,3} & & & & & & S_{4,3} \\
\uparrow & & & & & & \uparrow \\
S_{1,2} & & & & & & S_{4,2} \\
\uparrow & & & & & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}$$



Introduction (Extensions of PH, bipath)

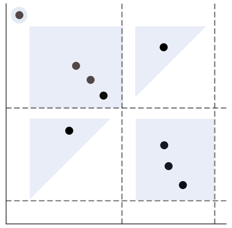
	Interval decomposability	○
(1)	Visualization (Bipath PD)	○
(2)	Algorithm (implementation)	○
(3)	Stability theorem for bipath PDs	○
	Inverse analysis	-
	Application	-

Implementation

Data → Bi-filtration → Bipath filtration → Bipath PD

$$\begin{array}{ccccccc} S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ S_{1,3} & \to & S_{2,3} & \to & S_{3,3} & \to & S_{4,3} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ S_{1,2} & \to & S_{2,2} & \to & S_{3,2} & \to & S_{4,2} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1} \end{array}$$

$$\begin{array}{ccccccc} S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\ \uparrow & & & & & & \uparrow \\ S_{1,3} & & & & & & S_{4,3} \\ \uparrow & & & & & & \uparrow \\ S_{1,2} & & & & & & S_{4,2} \\ \uparrow & & & & & & \uparrow \\ S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1} \end{array}$$



- Bipath persistence diagrams (Bipath PDs)
- Computational algorithm of bipath PDs
- A stability property of Bipath PDs.

- Bipath persistence diagrams (Bipath PDs)
- Computational algorithm of bipath PDs
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Recall: standard PDs

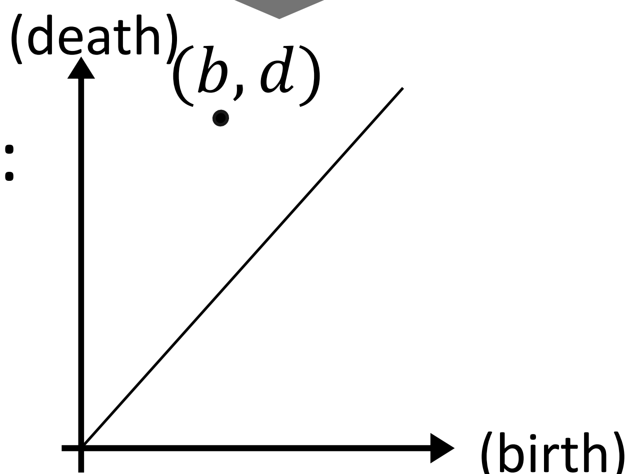
We need a correspondence of intervals and points in a plane to define persistence diagrams.

Underling poset: $\{1 \leq \cdots \leq b \leq \cdots \leq d \leq \cdots \leq n\}$

$\rotatebox{90}{$\subseteq$}$

Interval: $[b, d] \coloneqq \{b, b+1, \ldots, d\}$

Point in a plane:



Bipath Persistence

Definition Bipath poset

Let m and n be non-negative numbers. A *bipath poset* $B_{n,m}$ is a poset consisting of two totally ordered sets

$$1 \leq 2 \leq \cdots \leq n \text{ and } 1' \leq 2' \leq \cdots \leq m'$$

with the global minimum and the global maximum

$$\hat{0} \text{ and } \hat{1}.$$

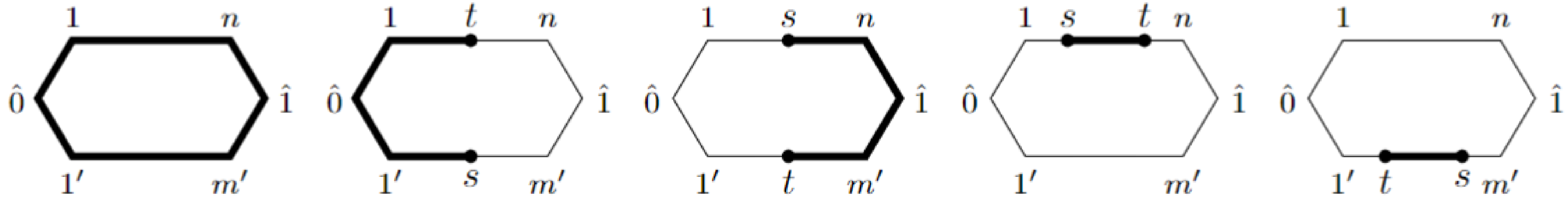
The Hasse diagram:

$$B_{n,m}: \quad \begin{tikzcd} & 1 \arrow[r] & 2 \arrow[r, "\leq"] & \cdots \arrow[r] & n \arrow[dr] & \\ \hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\ & 1' \arrow[r] & 2' \arrow[r, "\leq"] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd}$$

Bipath Persistence

Intervals in $B \coloneqq B_{n,m}$ are one of the following forms:

$\mathcal{B}$ (All)	$\mathcal{L}$ (Left)	$\mathcal{R}$ (Right)	$\mathcal{U}$ (Up)	$\mathcal{D}$ (Down)



$\hat{0} \in I$, $\hat{1} \in I$	$\hat{0} \in I$, $\hat{1} \notin I$	$\hat{0} \notin I$, $\hat{1} \in I$	$\hat{0} \notin I$, $\hat{1} \notin I$	$\hat{0} \notin I$, $\hat{1} \notin I$

・Each interval in B (except for B) is written by the pair $\langle s, t \rangle$ $(s, t \in B)$ by taking end points of the interval.

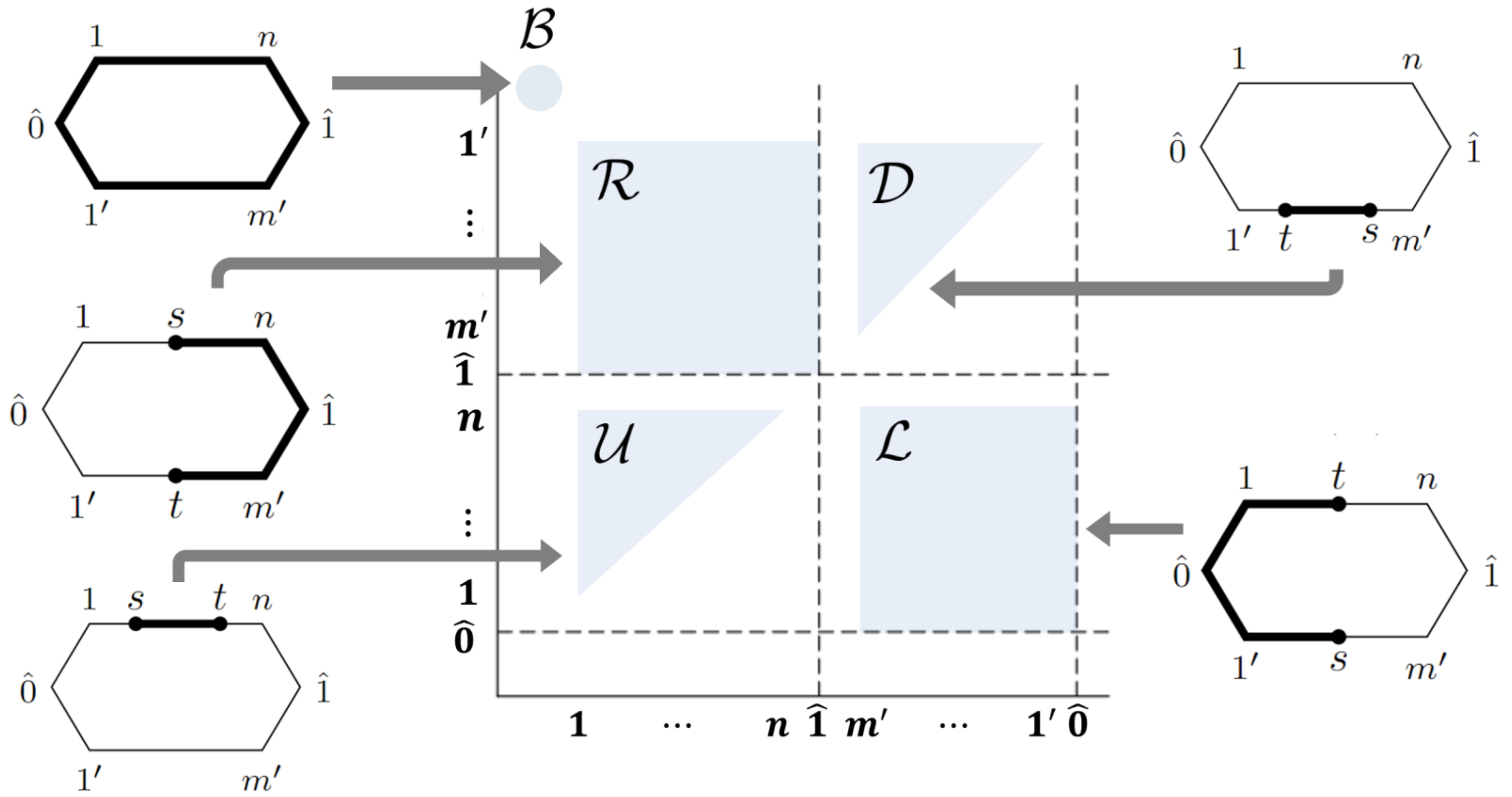
A correspondence of intervals and points

(1)Put elements of $B_{n,m}$ (in clockwise)
on the vertical and horizontal axes.

$$B_{n,m}\colon \begin{array}{ccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & \circlearrowright & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

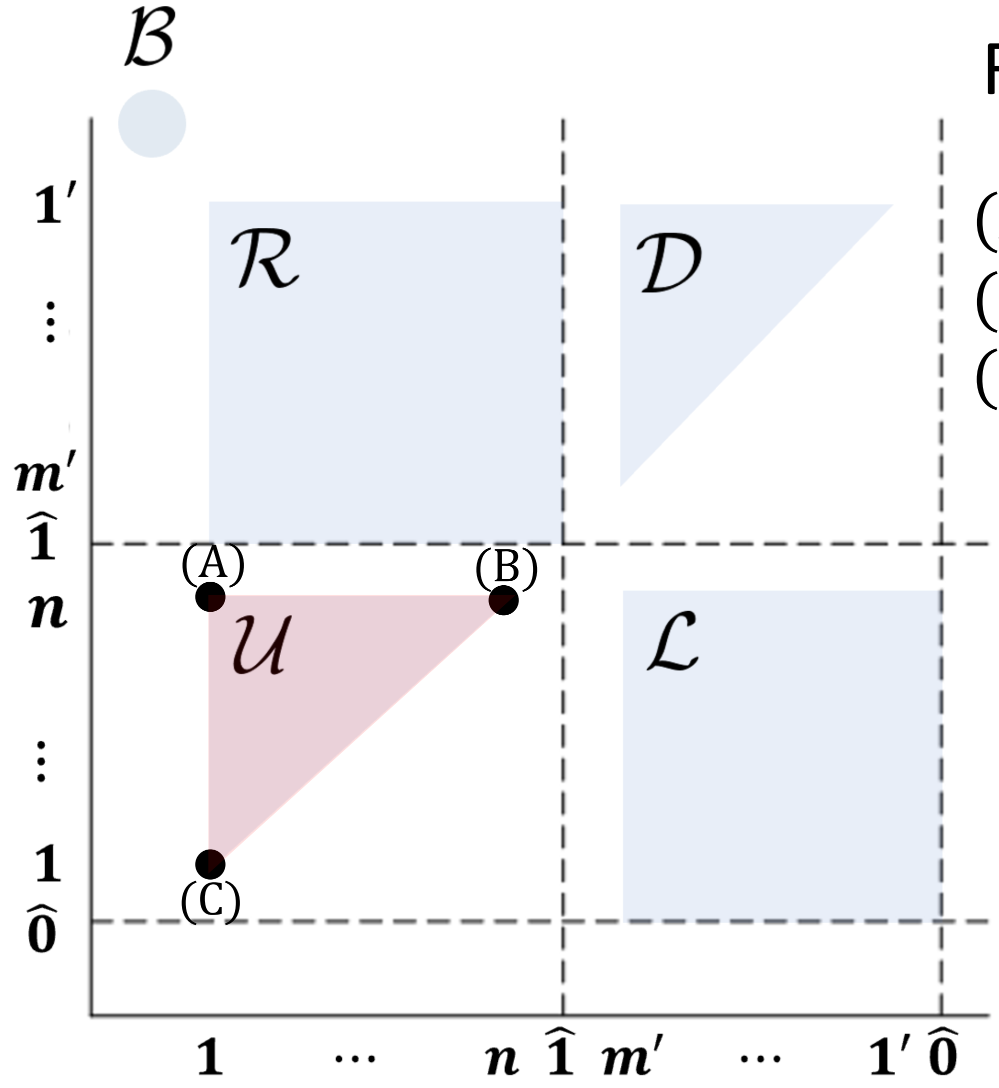
(2)Plot a point on the upper left region "$\mathcal{B}$" for the interval $B_{n,m}$.

(3)Plot a point (s, t) for the interval $\langle s, t \rangle$.



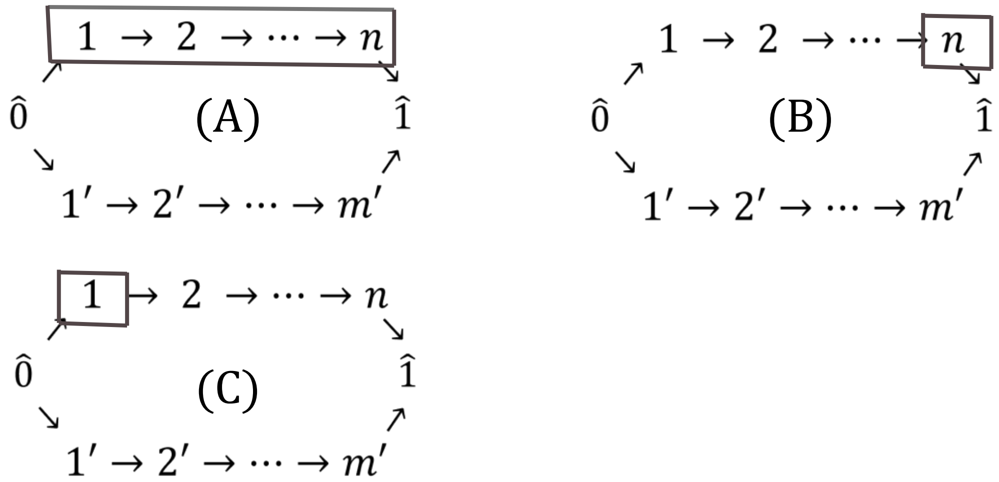
Bipath PD: multiset of points in the plane

Bipath PD: Region $\mathcal{U}$

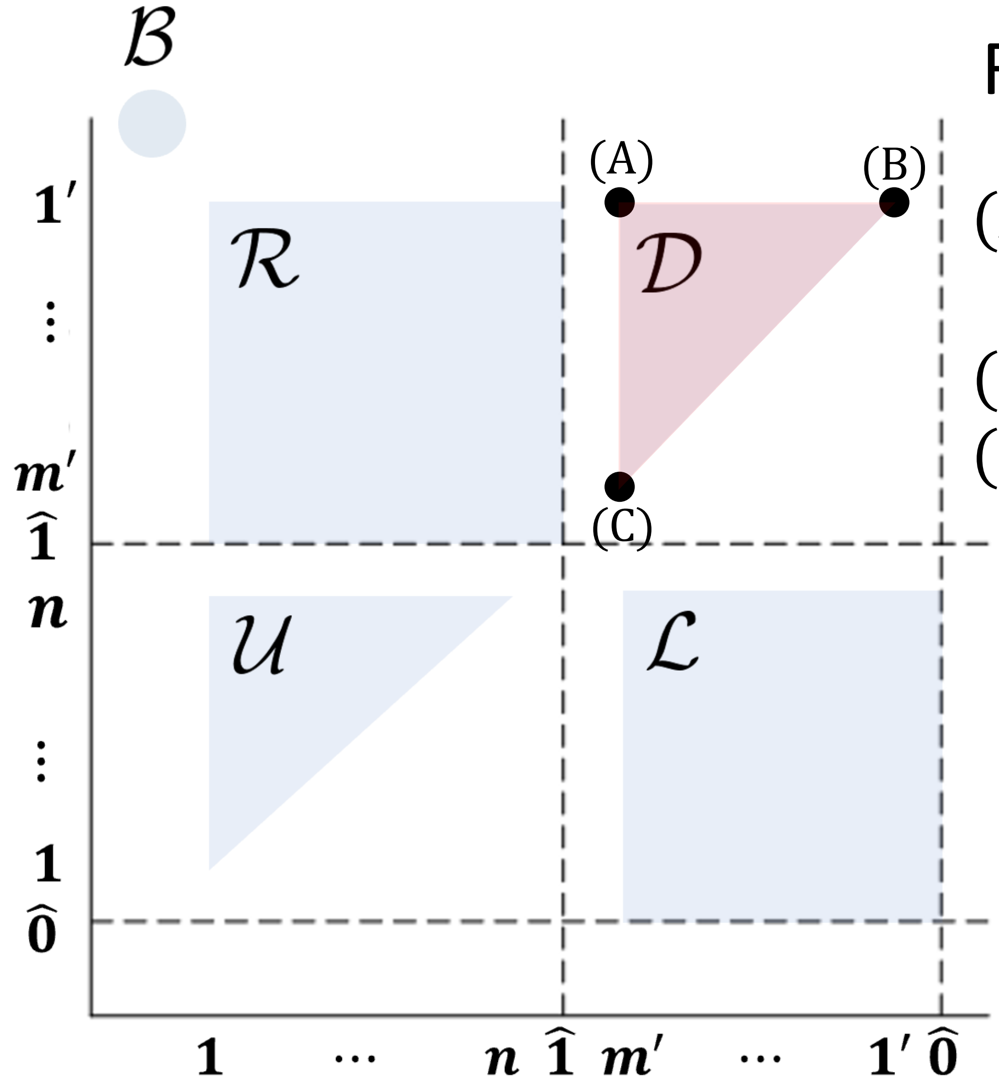


Region $\mathcal{U}$ is for intervals in $\mathcal{U}(B)$.

(A) $\cdots \langle 1, n \rangle = \{1, \ldots, n\}$. The longest interval in $\mathcal{U}(B)$.
(B) $\cdots \langle n, n \rangle = \{n\}$.
(C) $\cdots \langle 1,1 \rangle = \{1\}$.



Bipath PD: Region $\mathcal{D}$

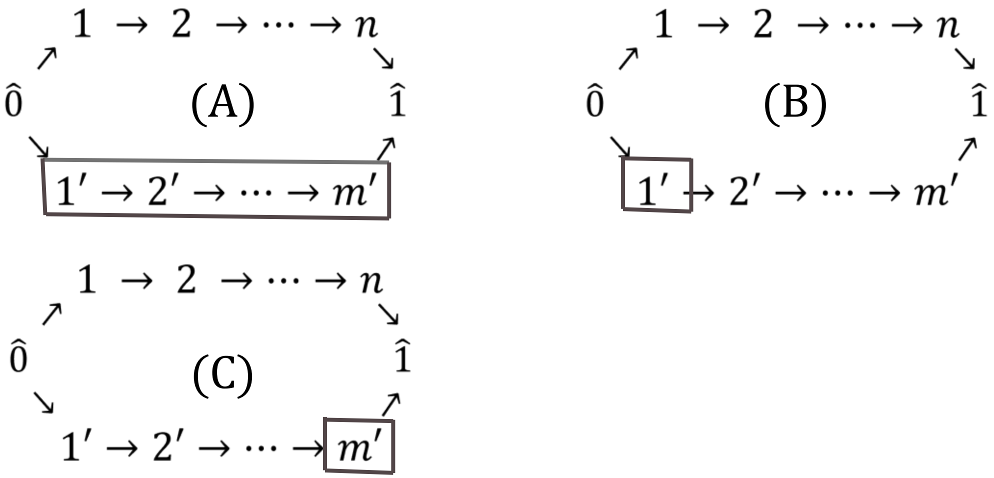


Region $\mathcal{D}$ is for intervals in $\mathcal{D}(B)$.

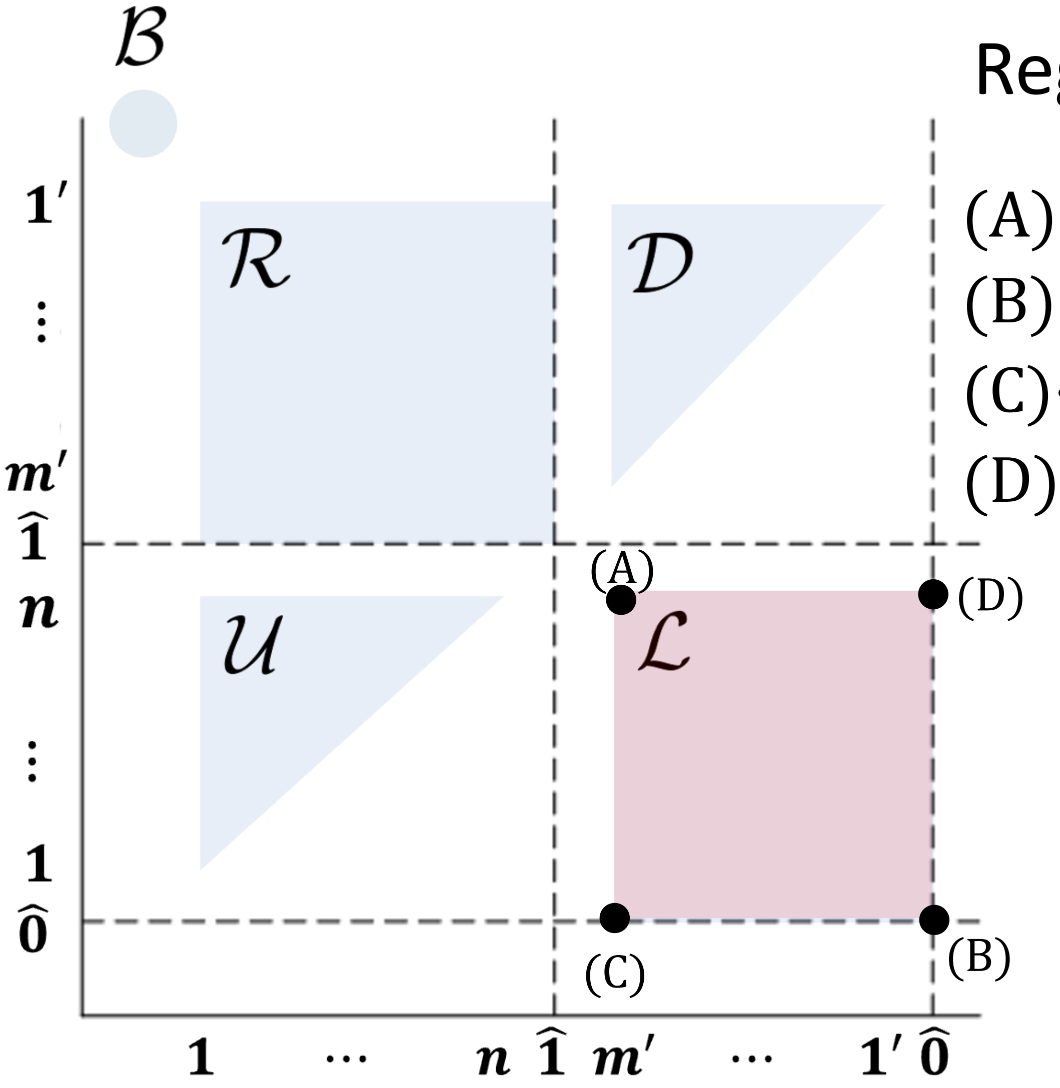
(A)$\cdots \langle m', 1' \rangle = \{m', \dots, 1'\}$. The longest interval in $\mathcal{D}(B)$.

(B) $\cdots \langle 1', 1' \rangle = \{1'\}$.

(C) $\cdots \langle m', m' \rangle = \{m'\}$.



Bipath PD: Region $\mathcal{L}$



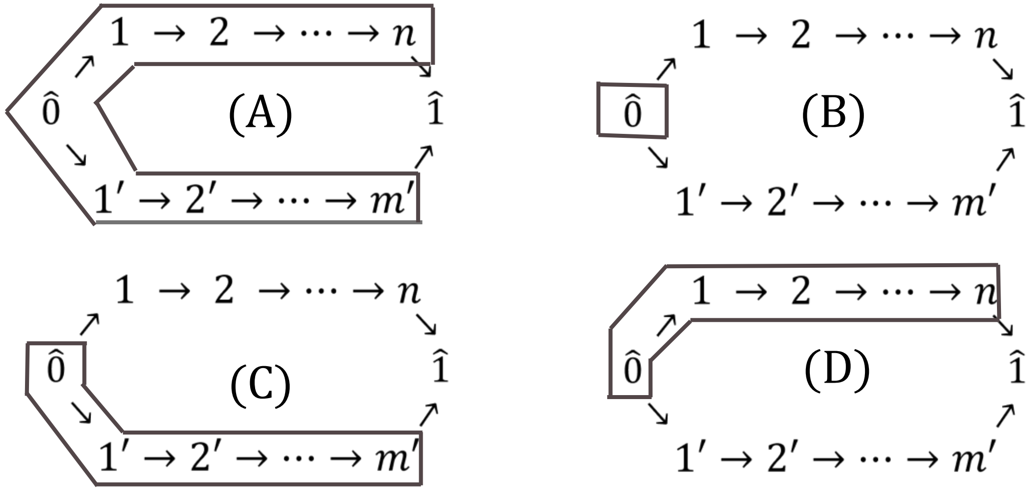
Region $\mathcal{L}$ is for intervals in $\mathcal{L}(B)$.

(A)$\cdots \langle m', n \rangle = B \setminus \{\hat{1}\}$. The longest interval in $\mathcal{L}(B)$.

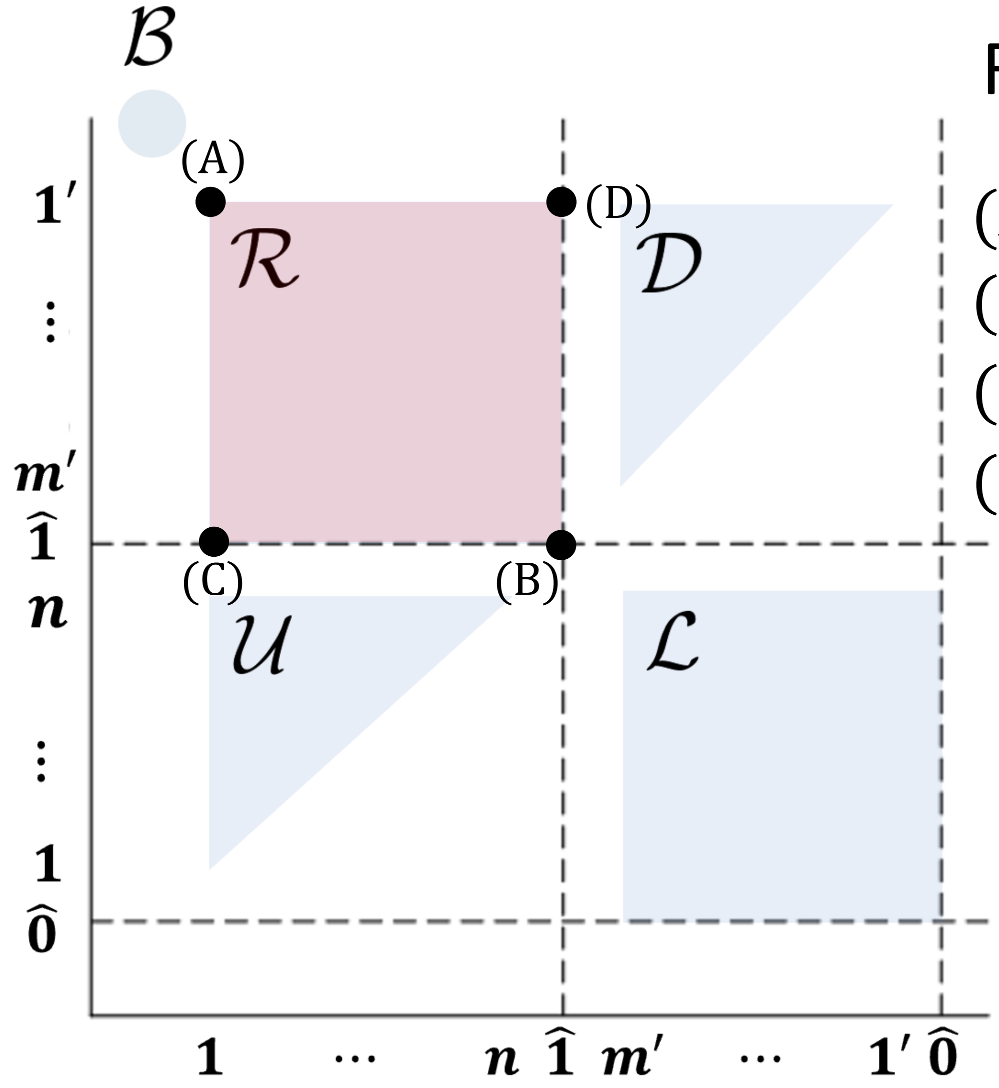
(B)$\cdots \langle \hat{0}, \hat{0} \rangle = \{\hat{0}\}$. The shortest interval in $\mathcal{L}(B)$

(C)$\cdots \langle m', \hat{0} \rangle = \{m', (m-1)', \dots, \hat{0}\}$.

(D)$\cdots \langle \hat{0}, n \rangle = \{\hat{0}, 1, \dots, n\}$.



Bipath PD: Region $\mathcal{R}$



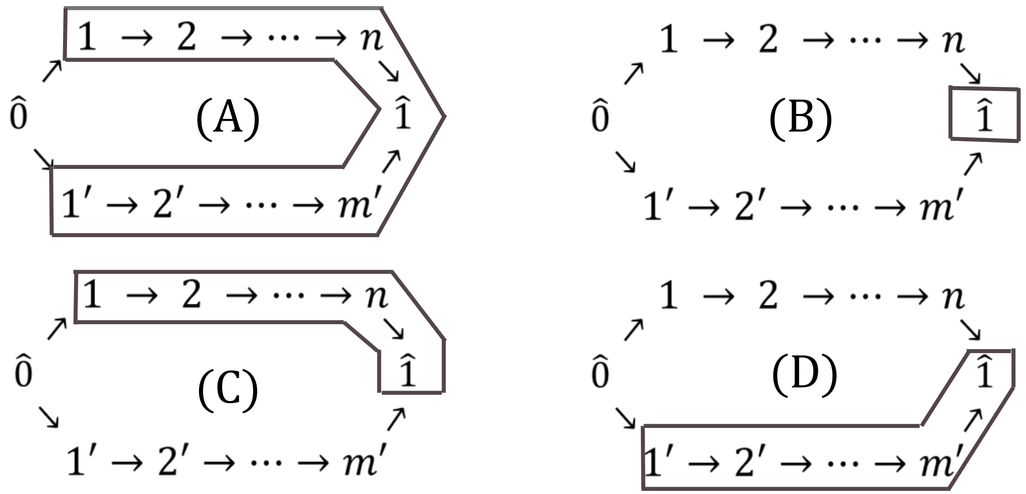
Region $\mathcal{R}$ is for intervals in $\mathcal{R}(B)$.

(A)⋯ $\langle 1,1' \rangle = B \setminus \{\hat{0}\}$. The longest interval in $\mathcal{R}(B)$.

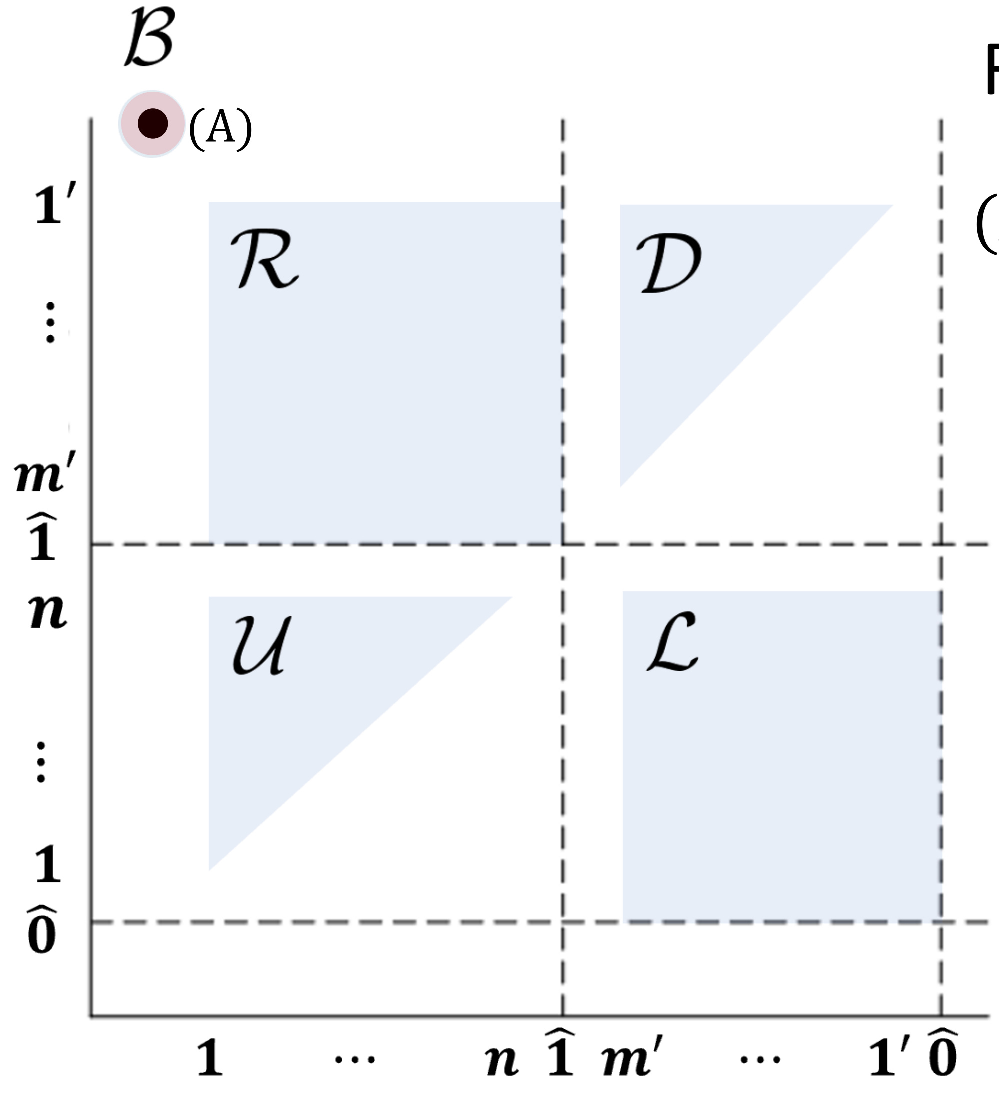
(B)⋯ $\langle \hat{1}, \hat{1} \rangle = \{\hat{1}\}$. The shortest interval in $\mathcal{R}(B)$.

(C)⋯ $\langle 1, \hat{1} \rangle = \{m', (m-1)', \dots, \hat{0}\}$.

(D)⋯ $\langle \hat{1}, 1' \rangle = \{\hat{0}, 1, \dots, n\}$.

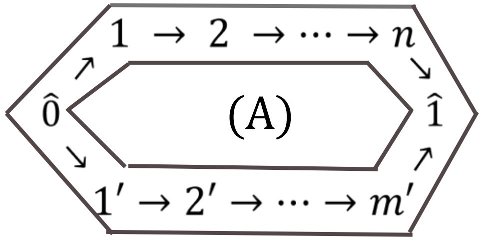


Bipath PD: Region $\mathcal{B}$



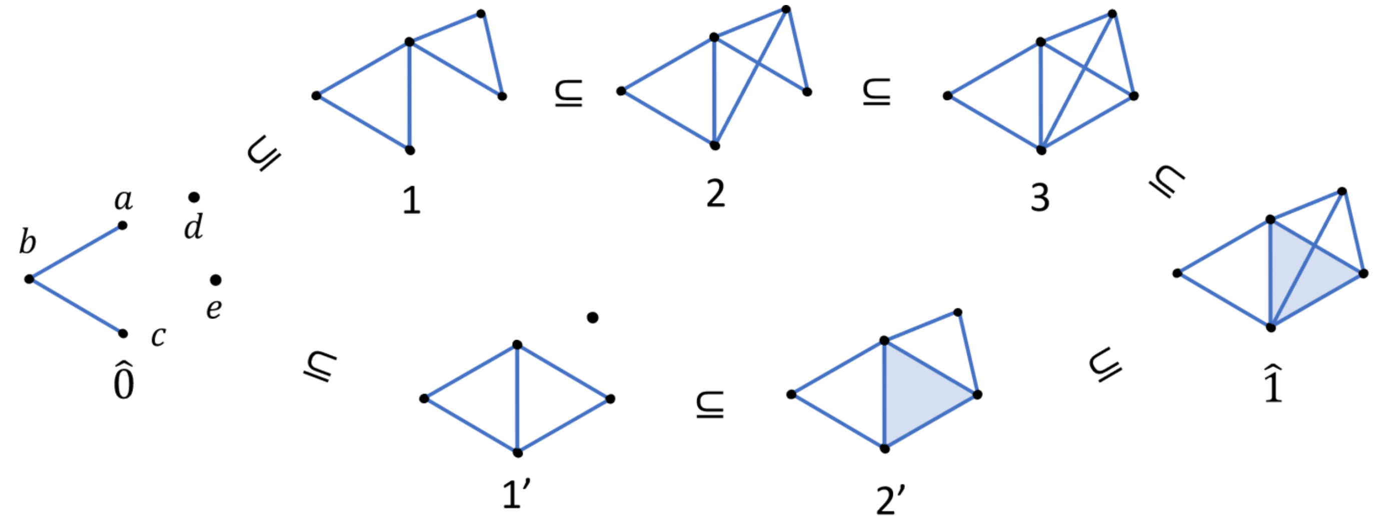
Region $\mathcal{B}$ is for intervals in $\mathcal{B}(B)(=\{B\})$.

(A)$\cdots B$.



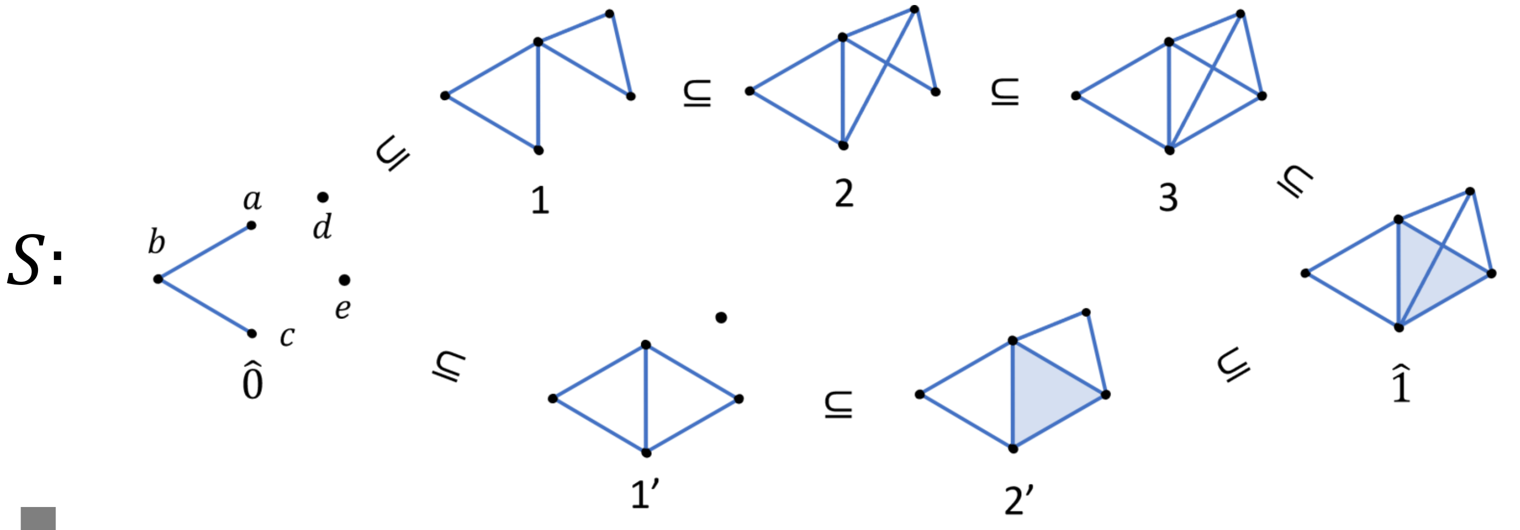
Examples of bipath PD

S:

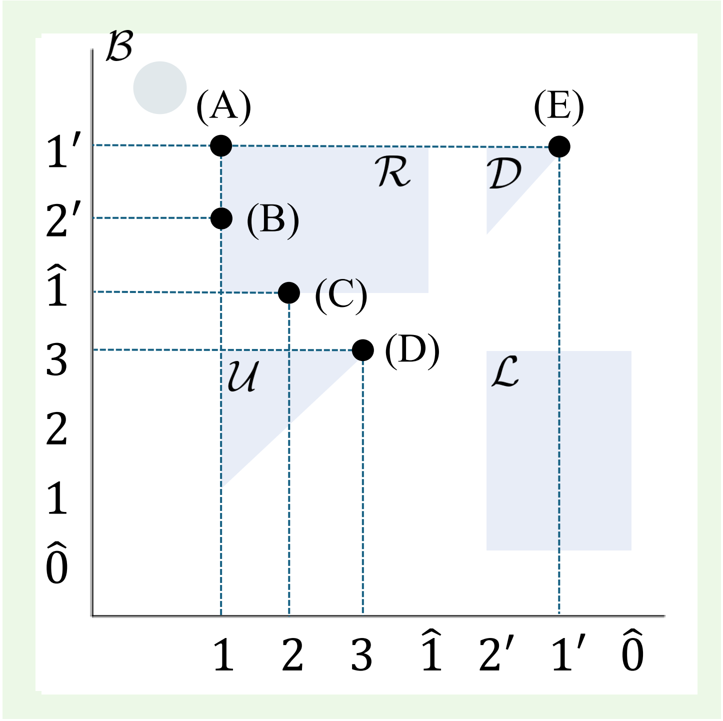


- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

Examples of bipath PD



1st.

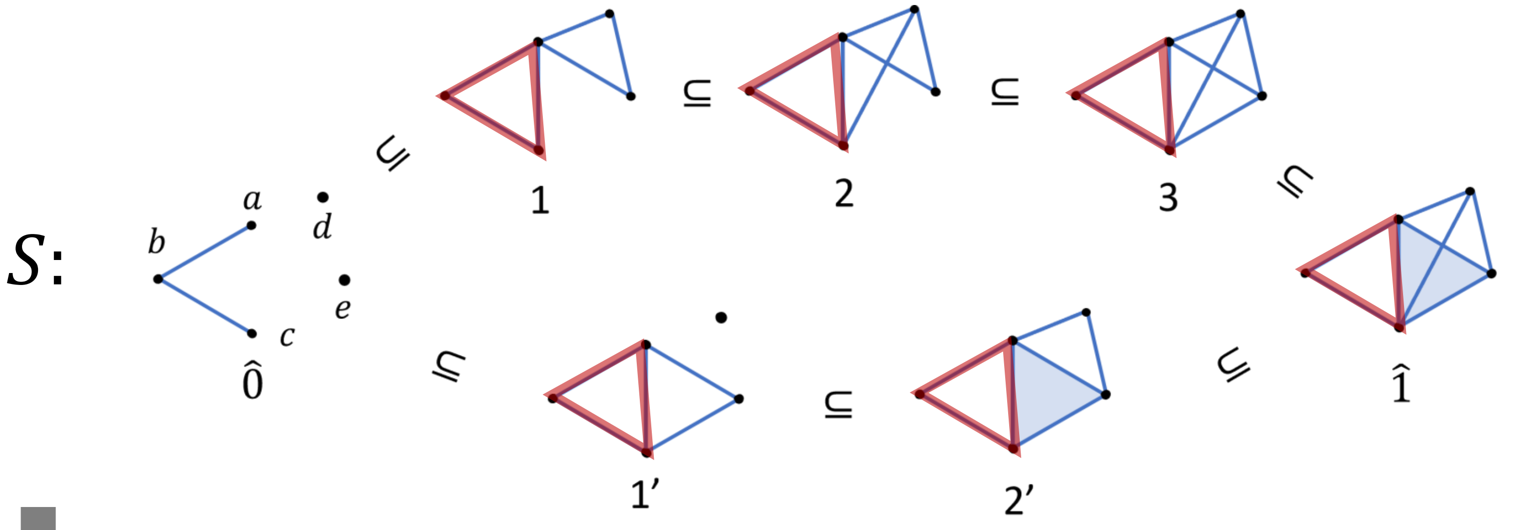


- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

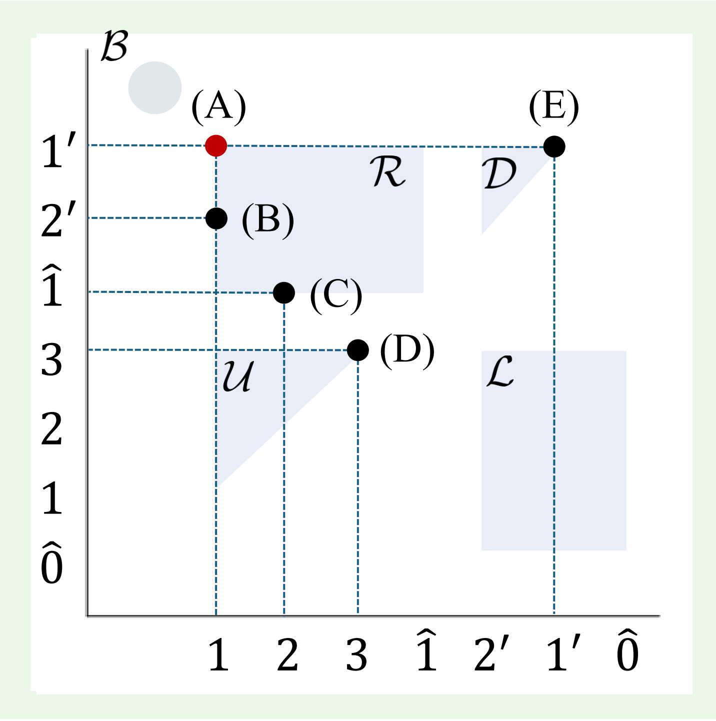
$$\mathcal{B}(H_1(S;k)) = \{\{1,2,3,\hat{1},2',1'\},\{1,2,3,\hat{1},2'\},\{2,3,\hat{1}\},\{3\},\ \{1'\}\}$$

$$= \{\underset{\text{(A)}}{\langle 1,1'\rangle}, \underset{\text{(B)}}{\langle 1,2'\rangle}, \underset{\text{(C)}}{\langle 2,\hat{1}\rangle}, \underset{\text{(D)}}{\langle 3,3\rangle}, \underset{\text{(E)}}{\langle 1',1'\rangle}\}.$$

Examples of bipath PD



1st.



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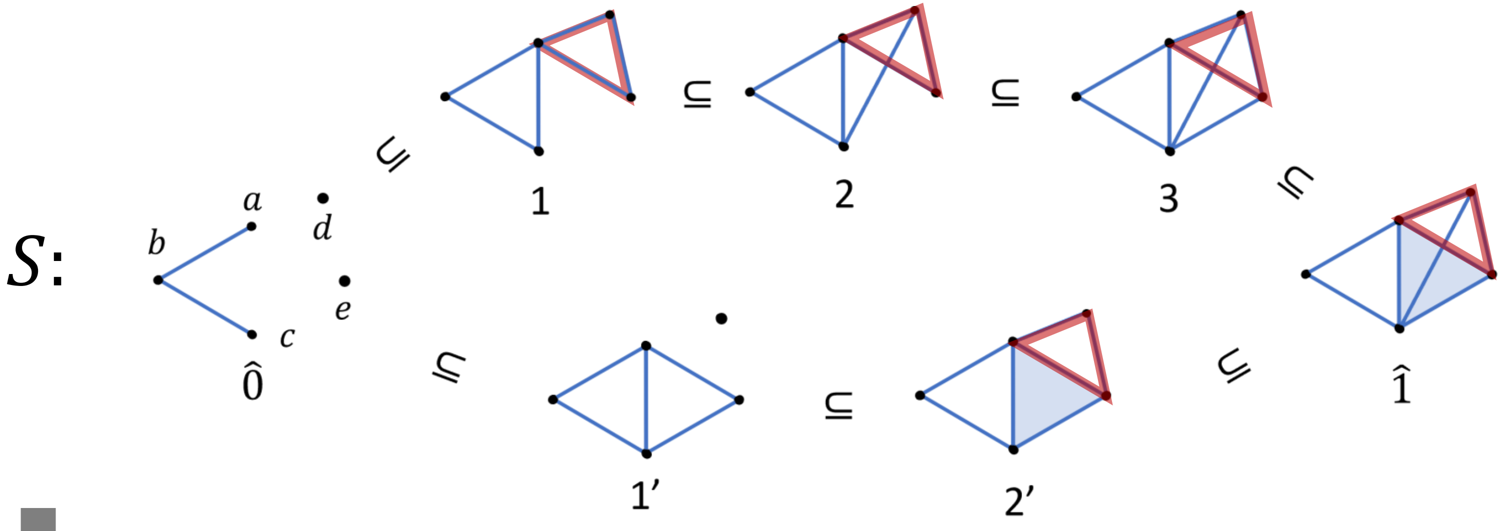
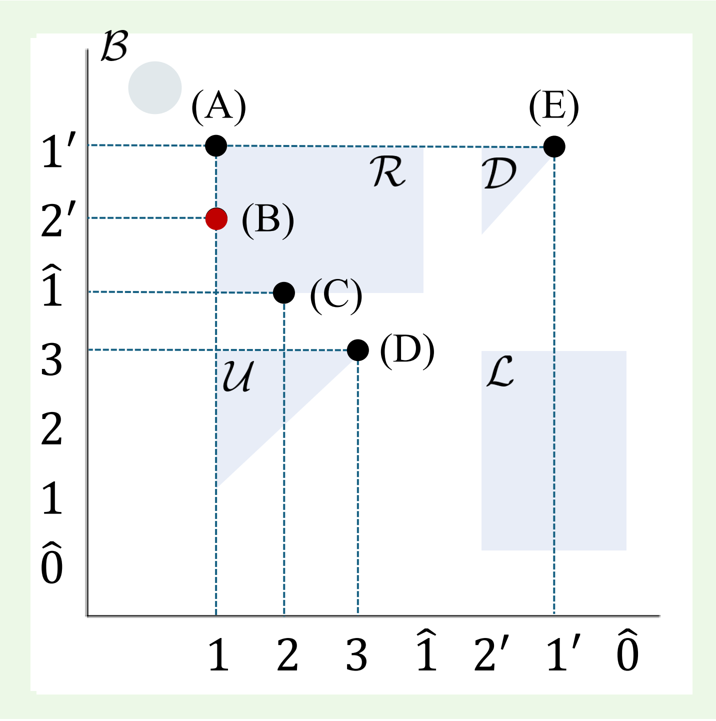
$$= \{\underset{\text{(A)}}{\boxed{\langle 1,1'\rangle}}, \underset{\text{(B)}}{\langle 1,2'\rangle}, \underset{\text{(C)}}{\langle 2,\hat{1}\rangle}, \underset{\text{(D)}}{\langle 3,3\rangle}, \underset{\text{(E)}}{\langle 1',1'\rangle}\}.$$

Examples of bipath PD

1st.





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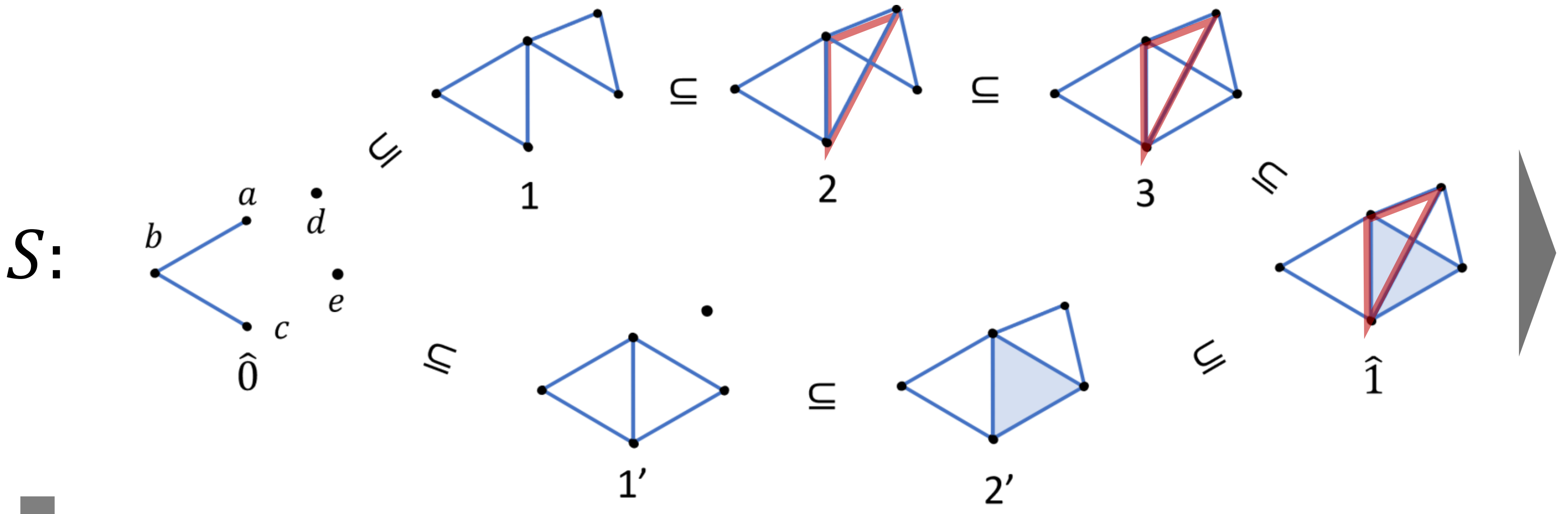
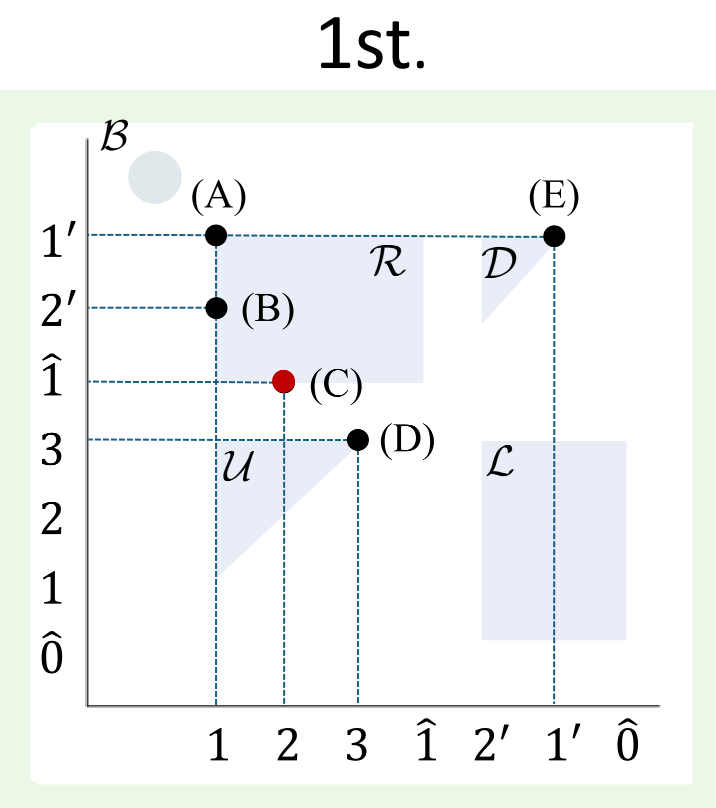
Examples of bipath PD





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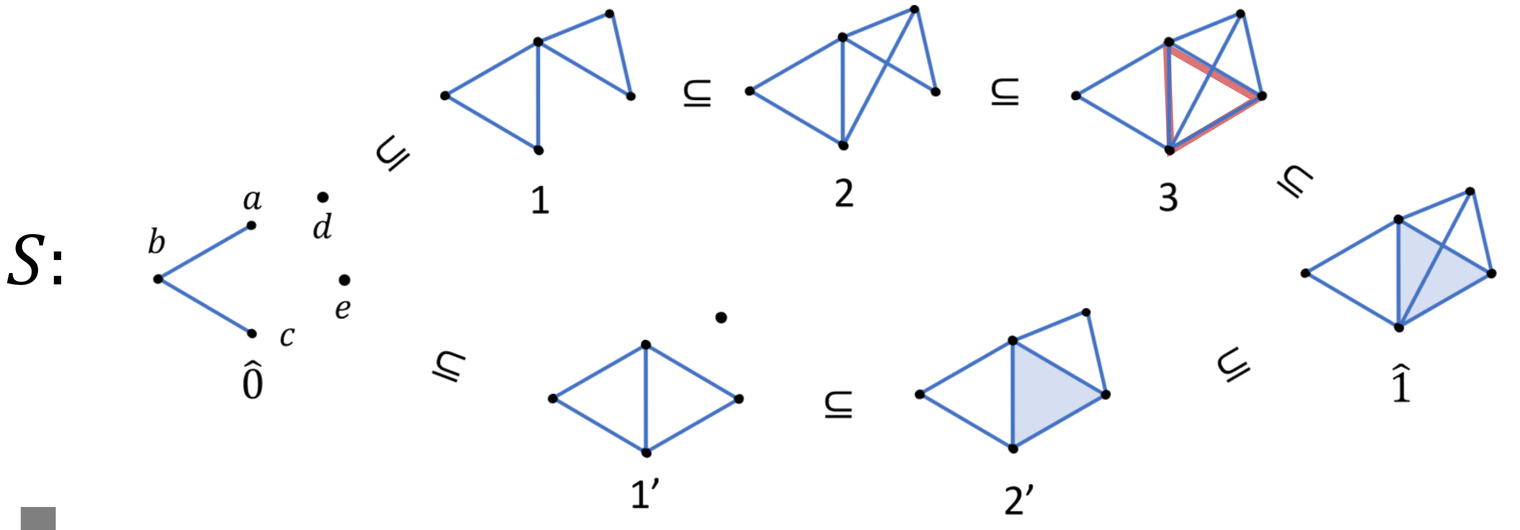
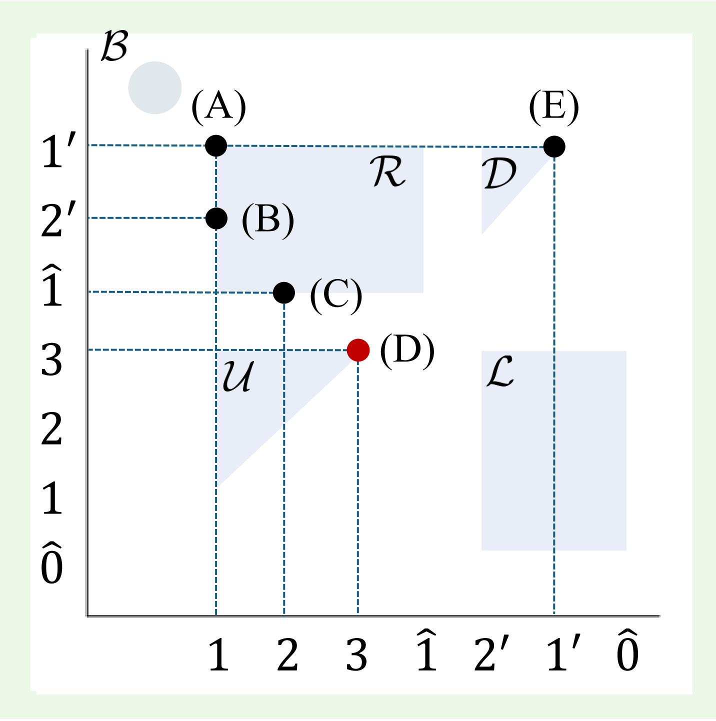
Examples of bipath PD



1st.



- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

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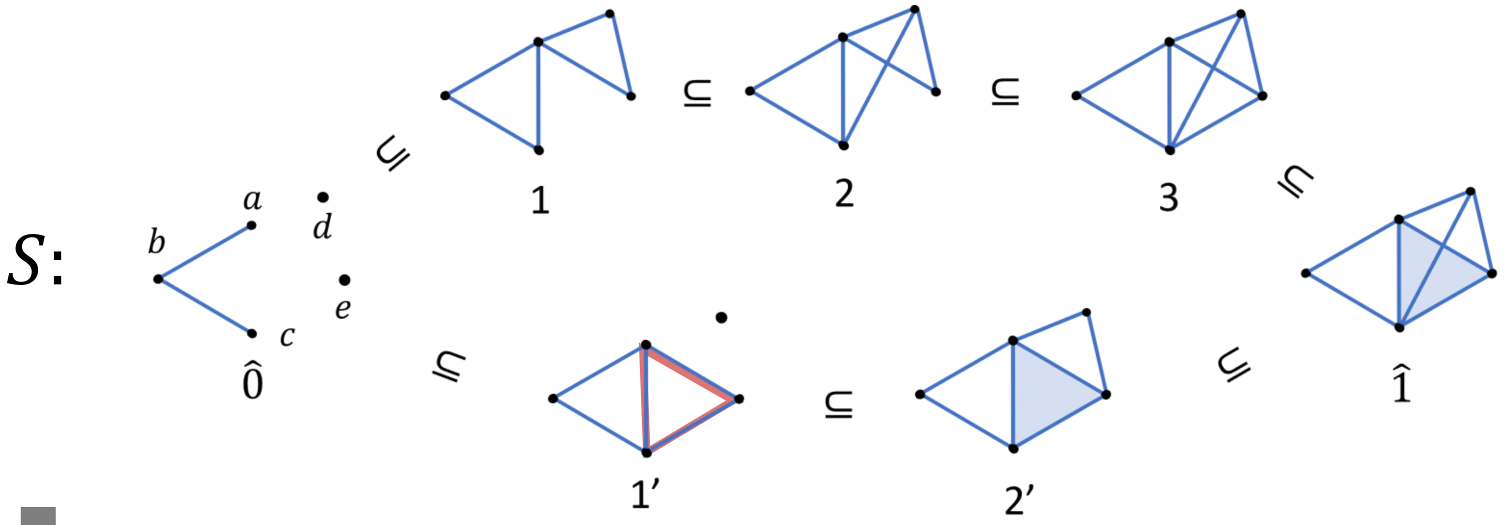
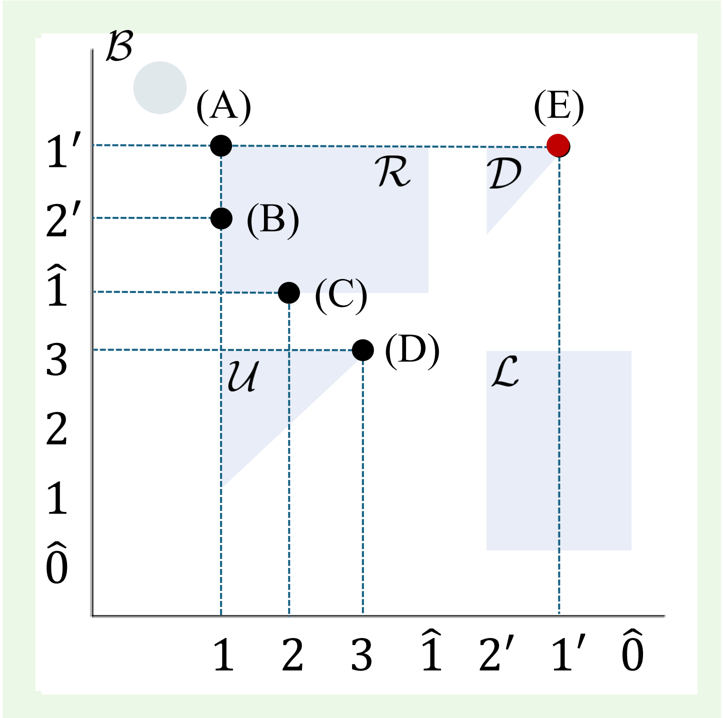
Examples of bipath PD

1st.





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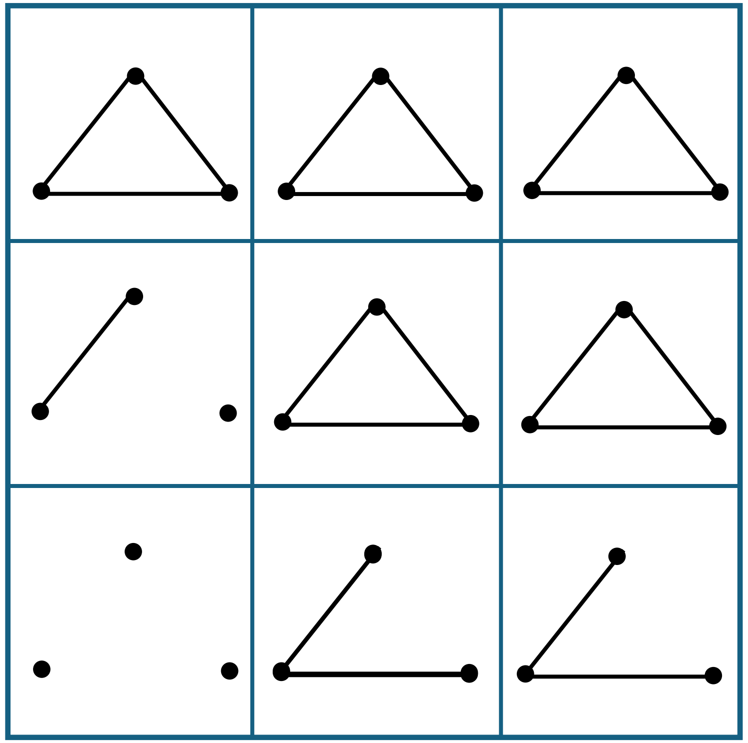
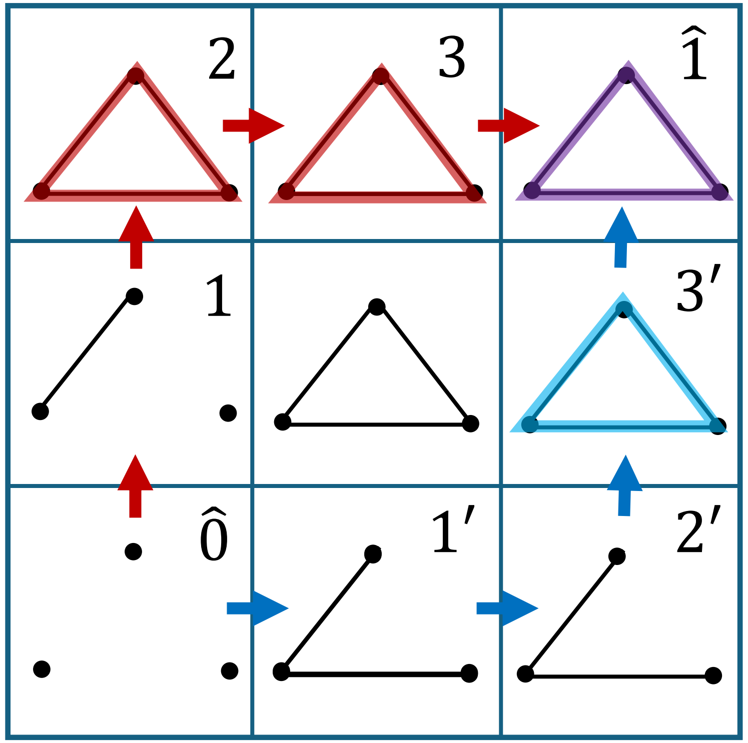
Examples of bipath PD



Examples of bipath PD

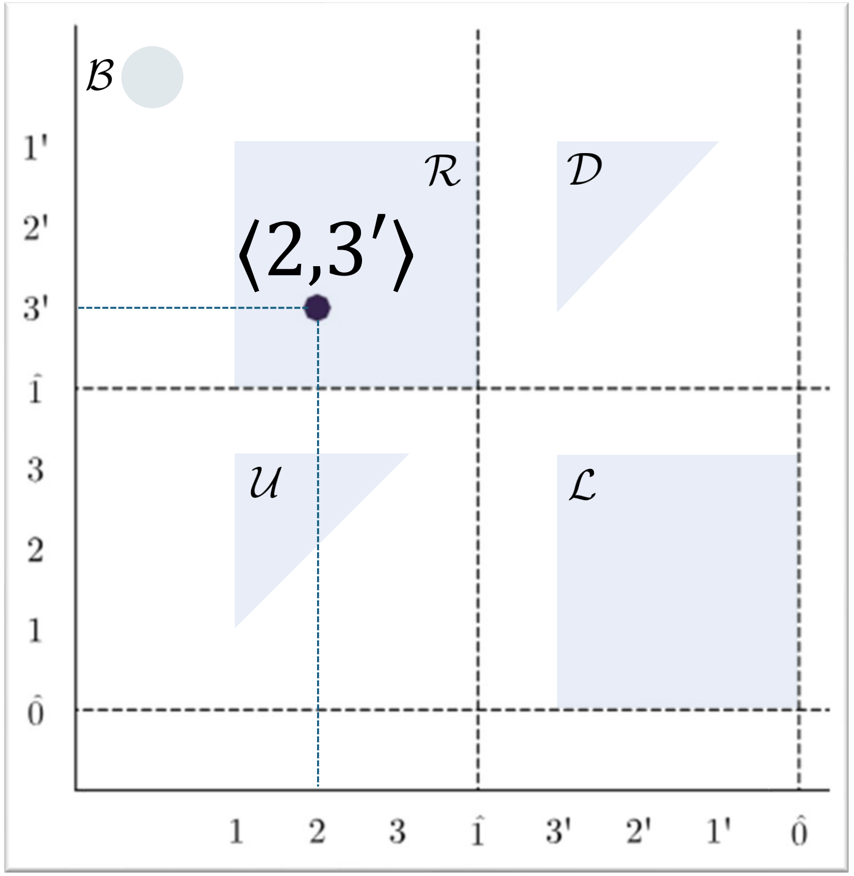


:Upper path

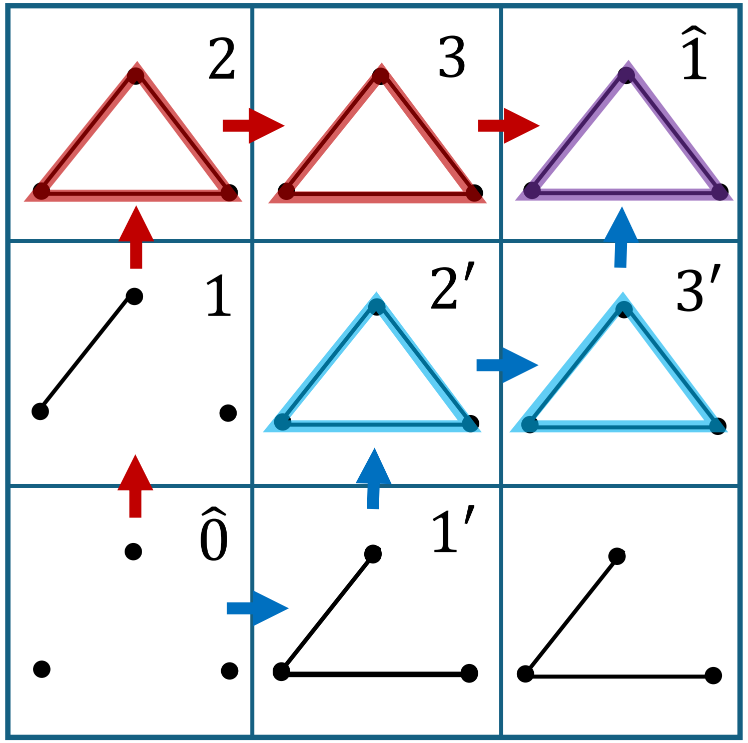
:Lower path

Get bipath PH.
$(k = \mathbb{F}_2)$

1st.



Examples of bipath PD



:Upper path

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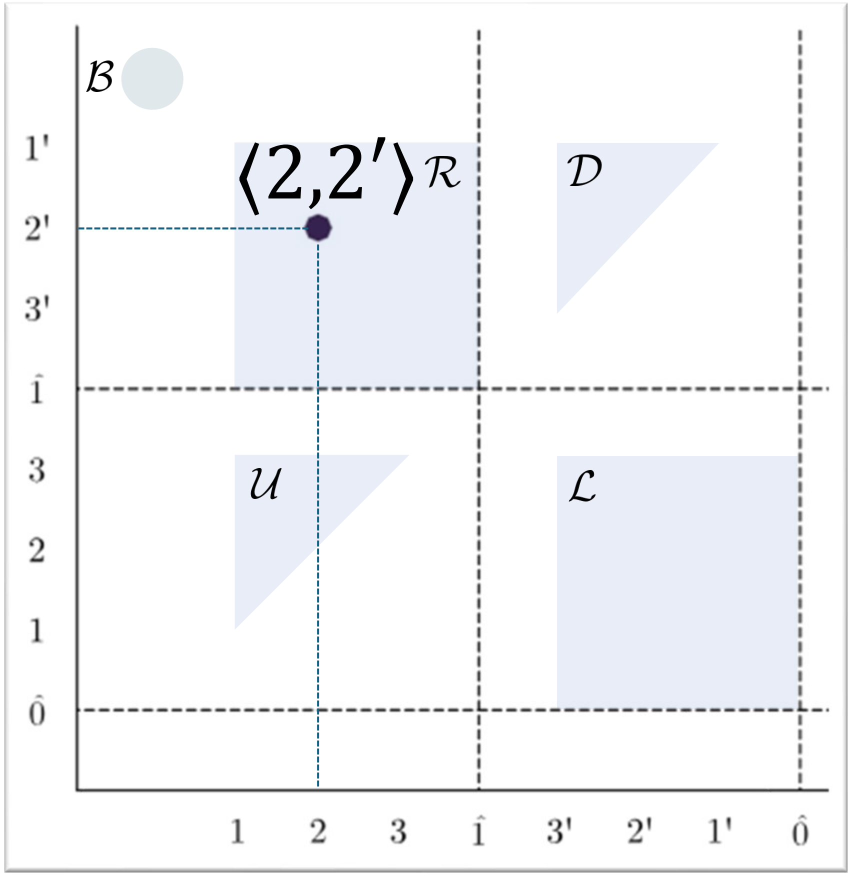
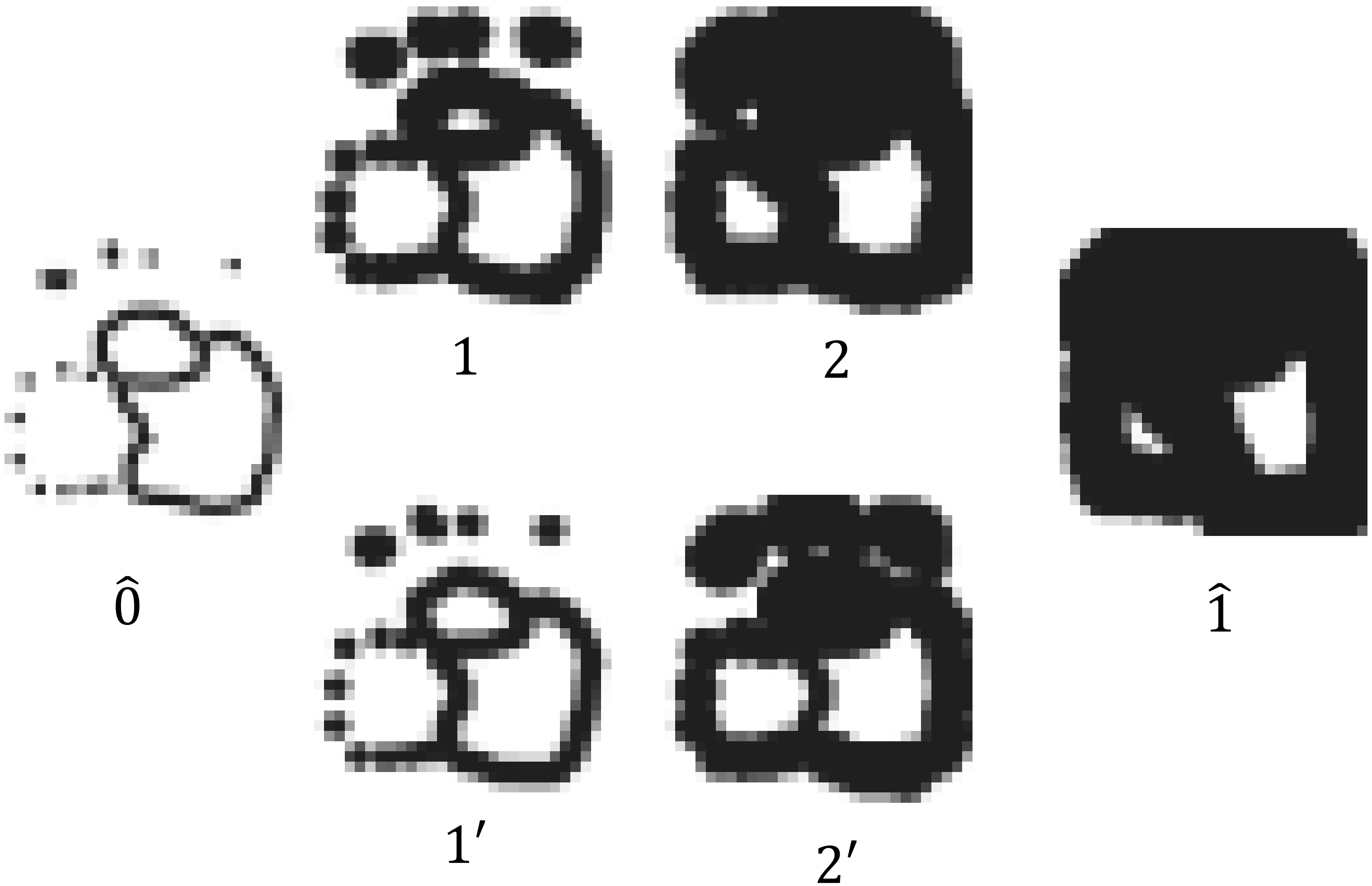
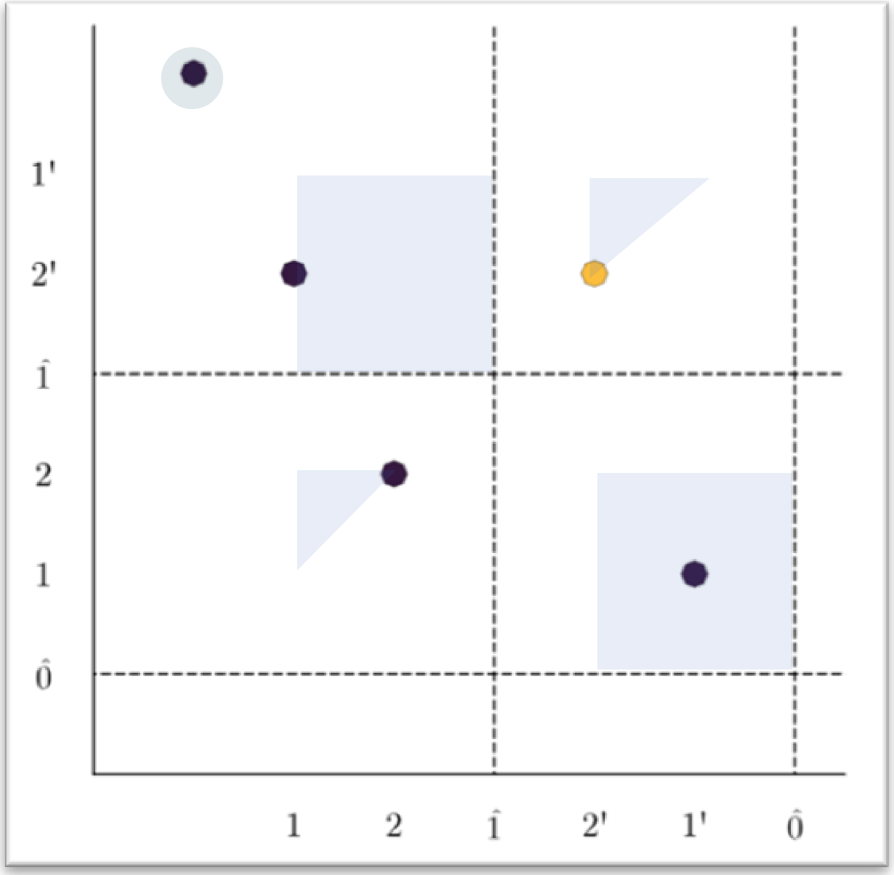


Examples of bipath PD



Bipath filtration of image data (30 × 30pixel)

1st.

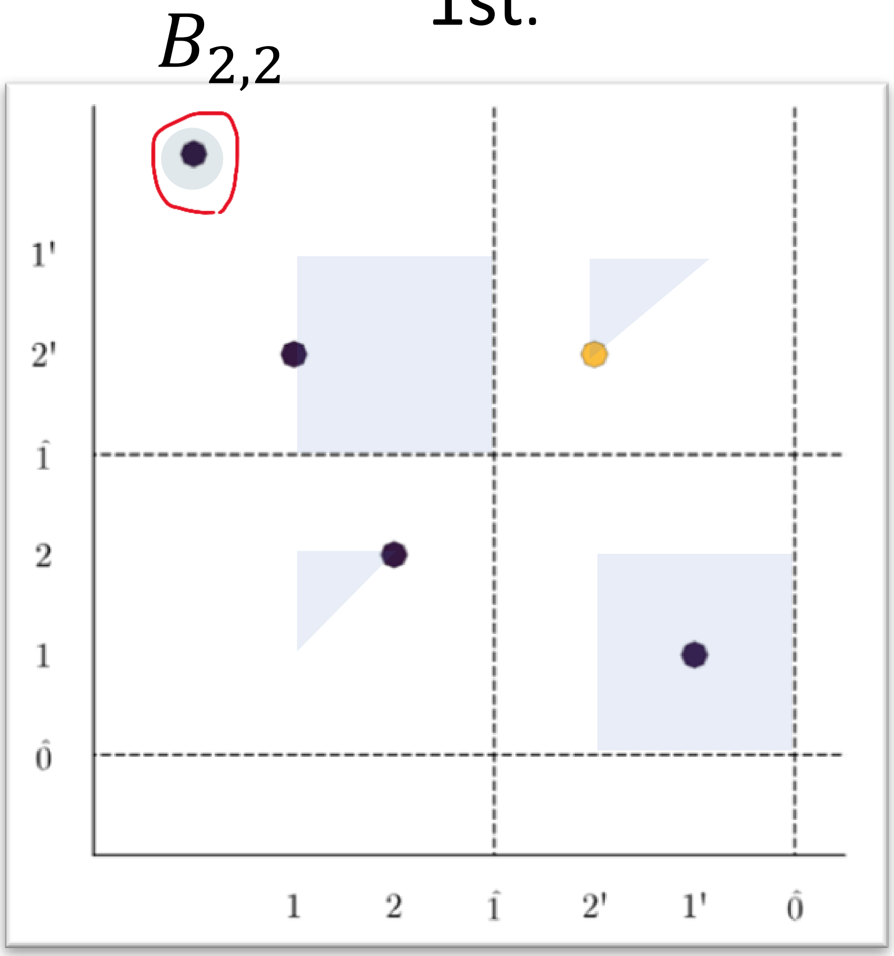


Colors are changed by the multiplicity of intervals.

Examples of bipath PD

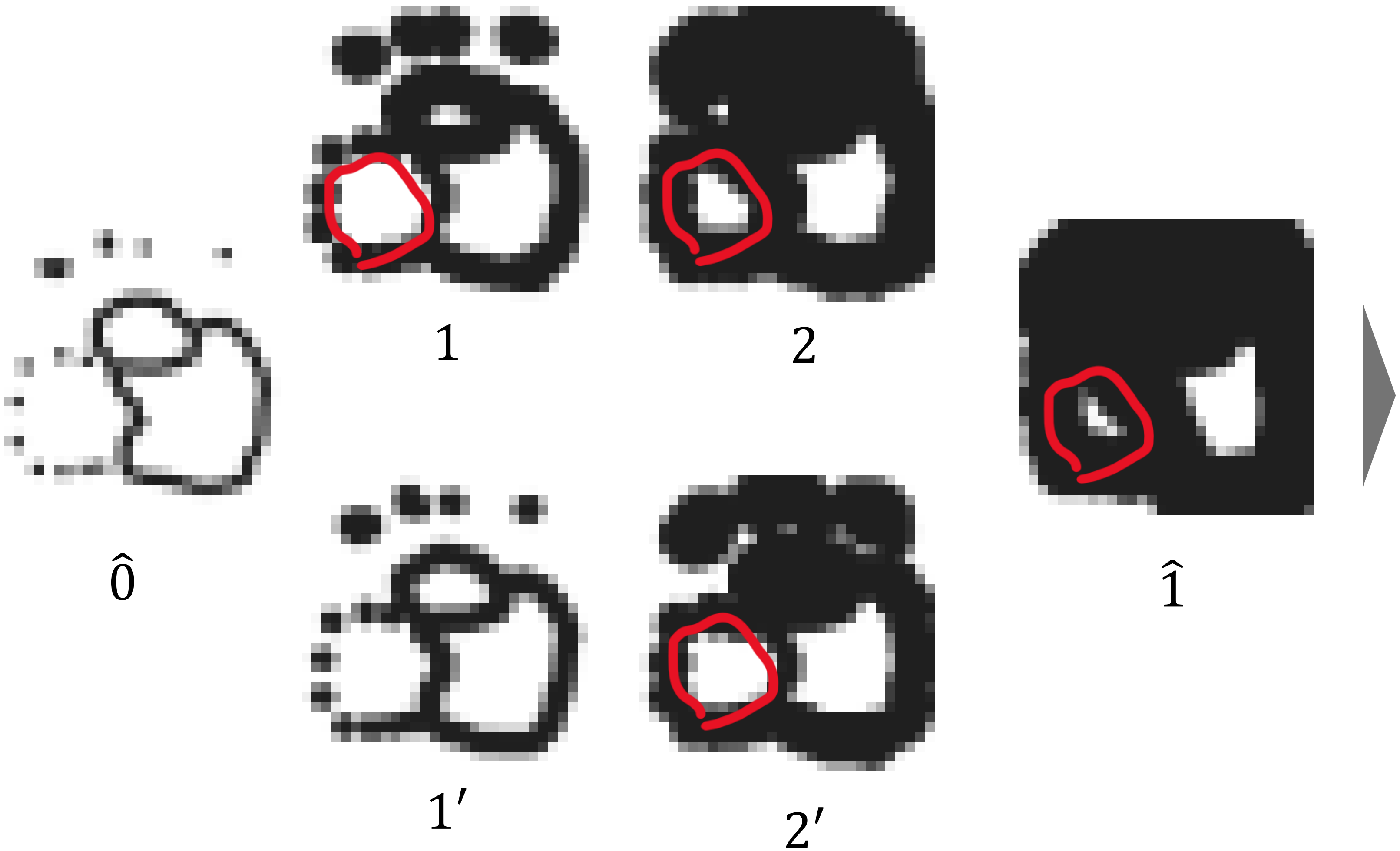
Bipath filtration of image data (30 × 30pixel)

1st.



Colors are changed by the multiplicity of intervals.

Examples of bipath PD



Bipath filtration of image data (30 × 30pixel)

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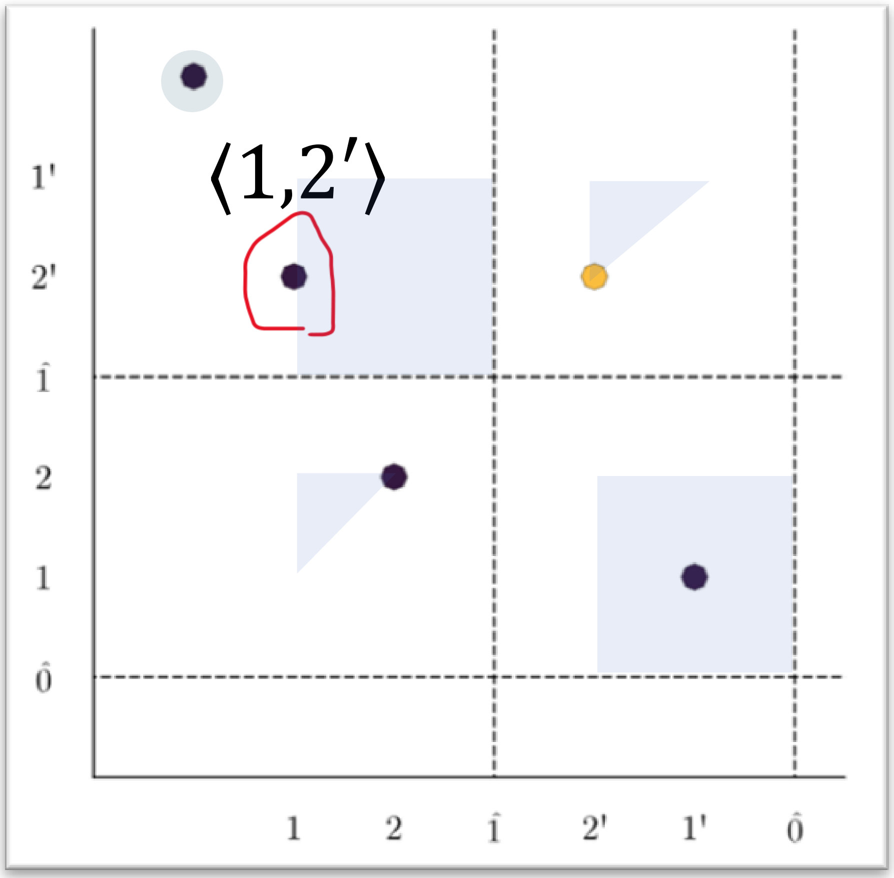
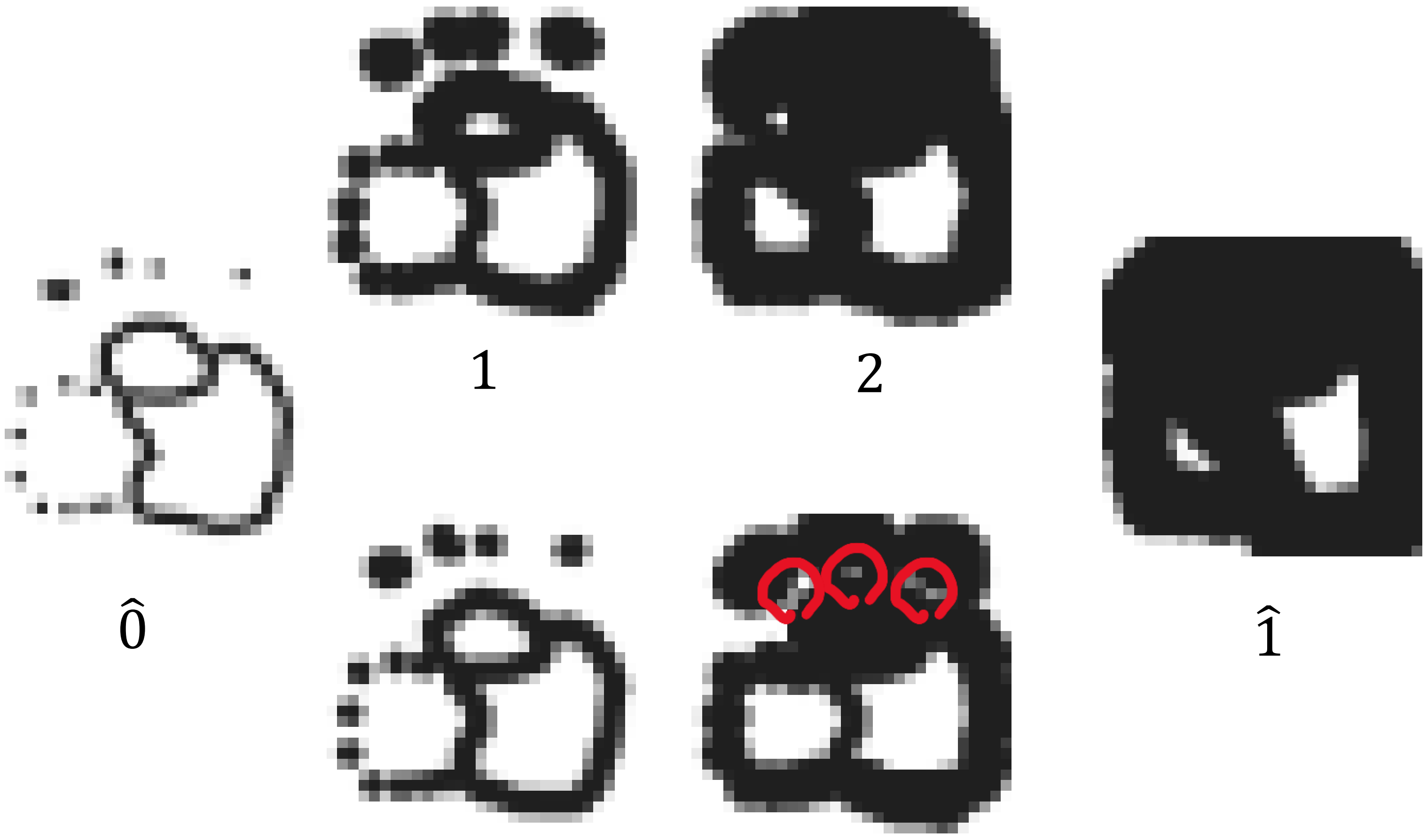


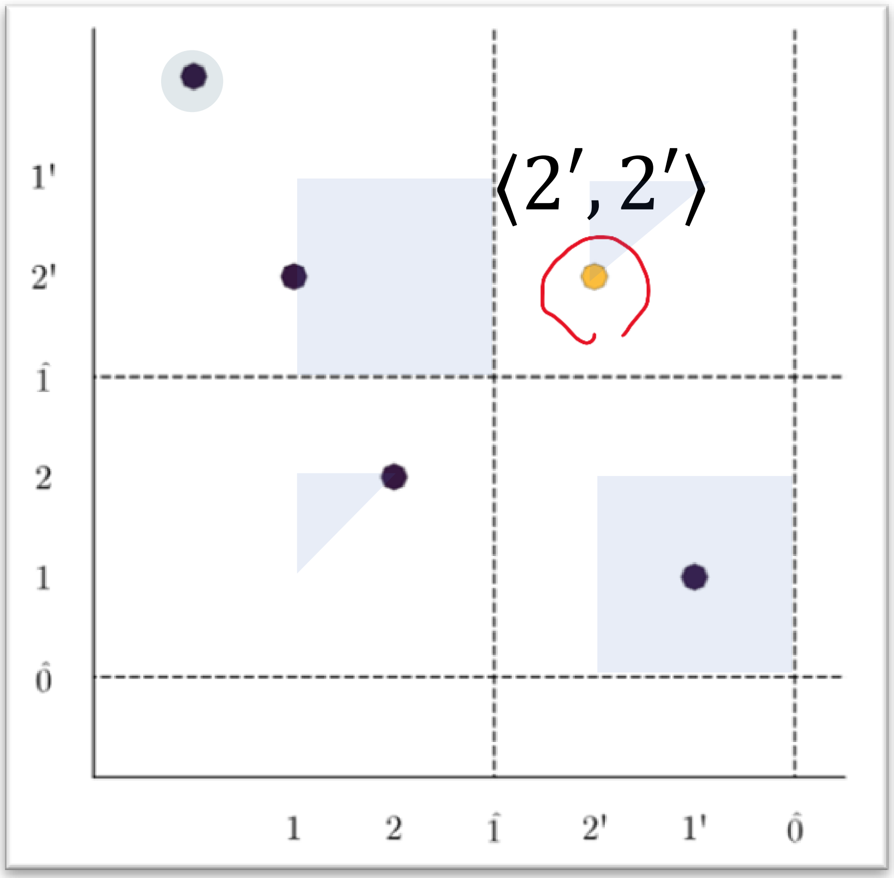
Colors are changed by the multiplicity of intervals.

Examples of bipath PD



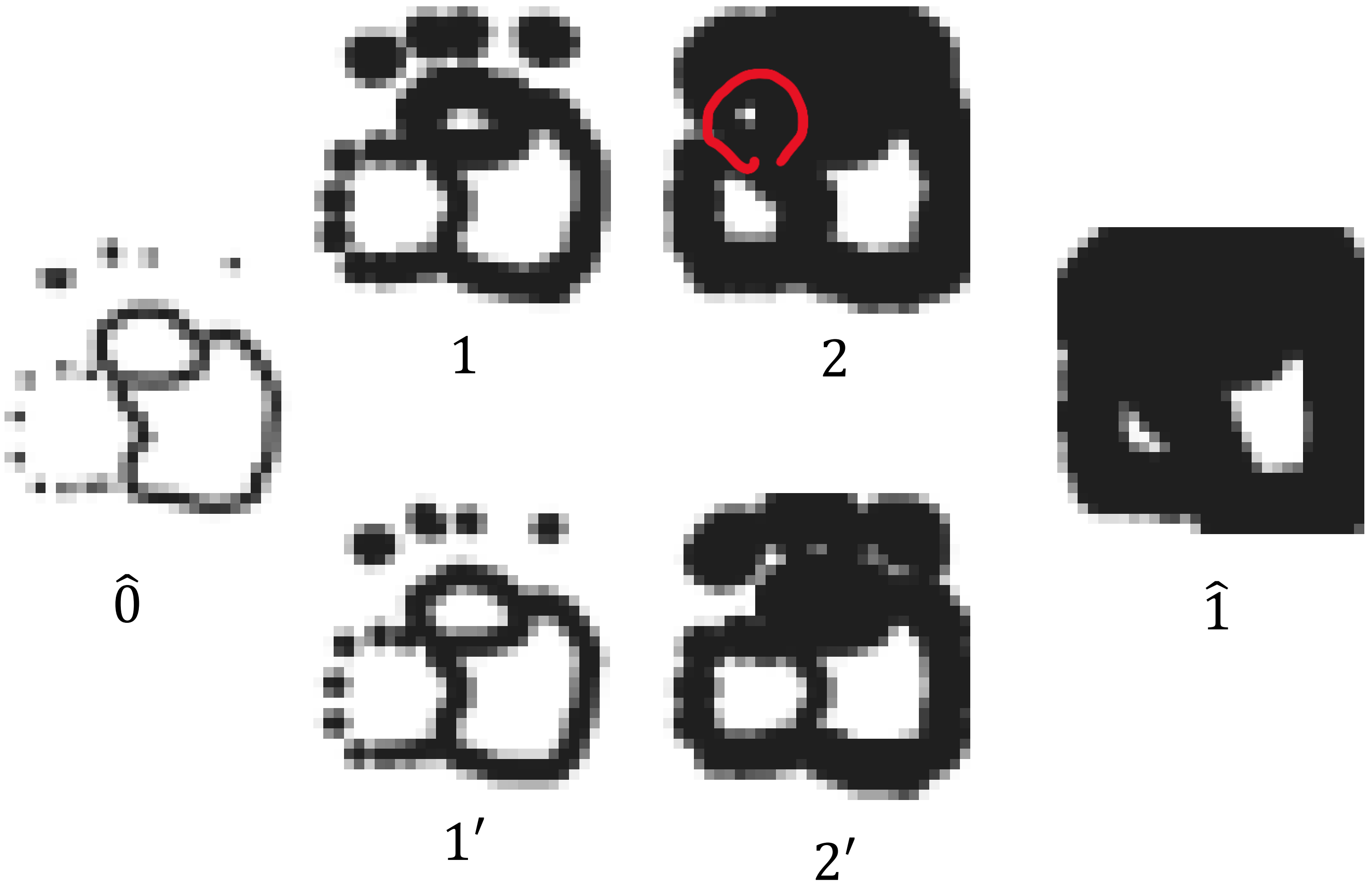
Bipath filtration of image data (30 × 30pixel)

1st.



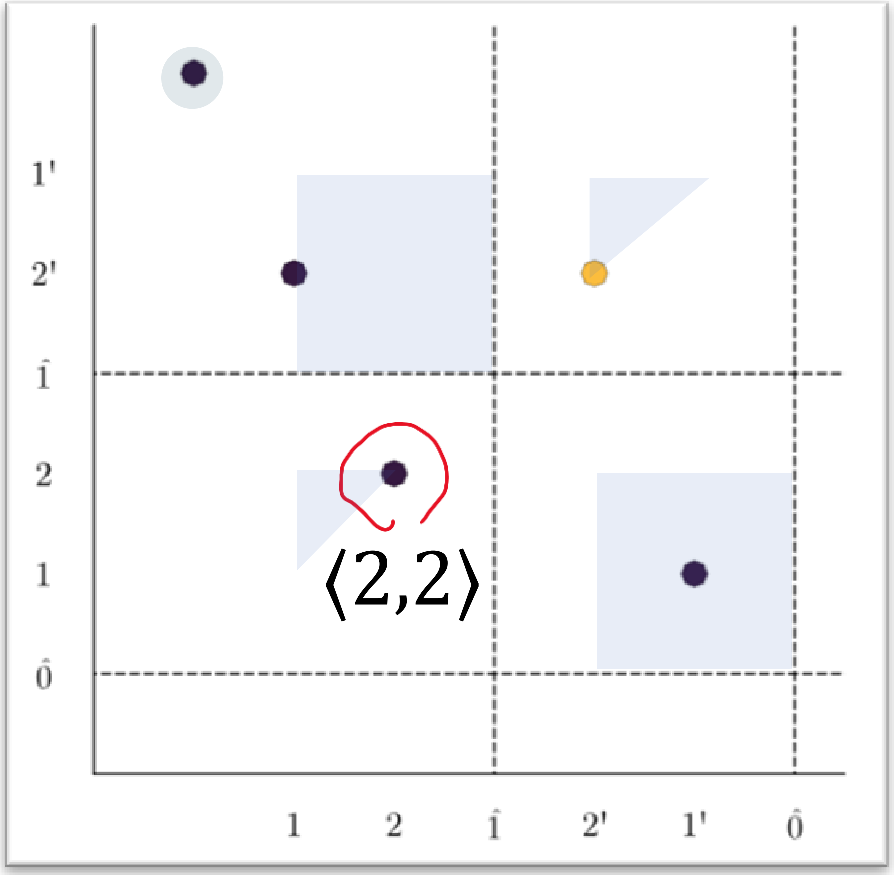
Colors are changed by the multiplicity of intervals.

Examples of bipath PD



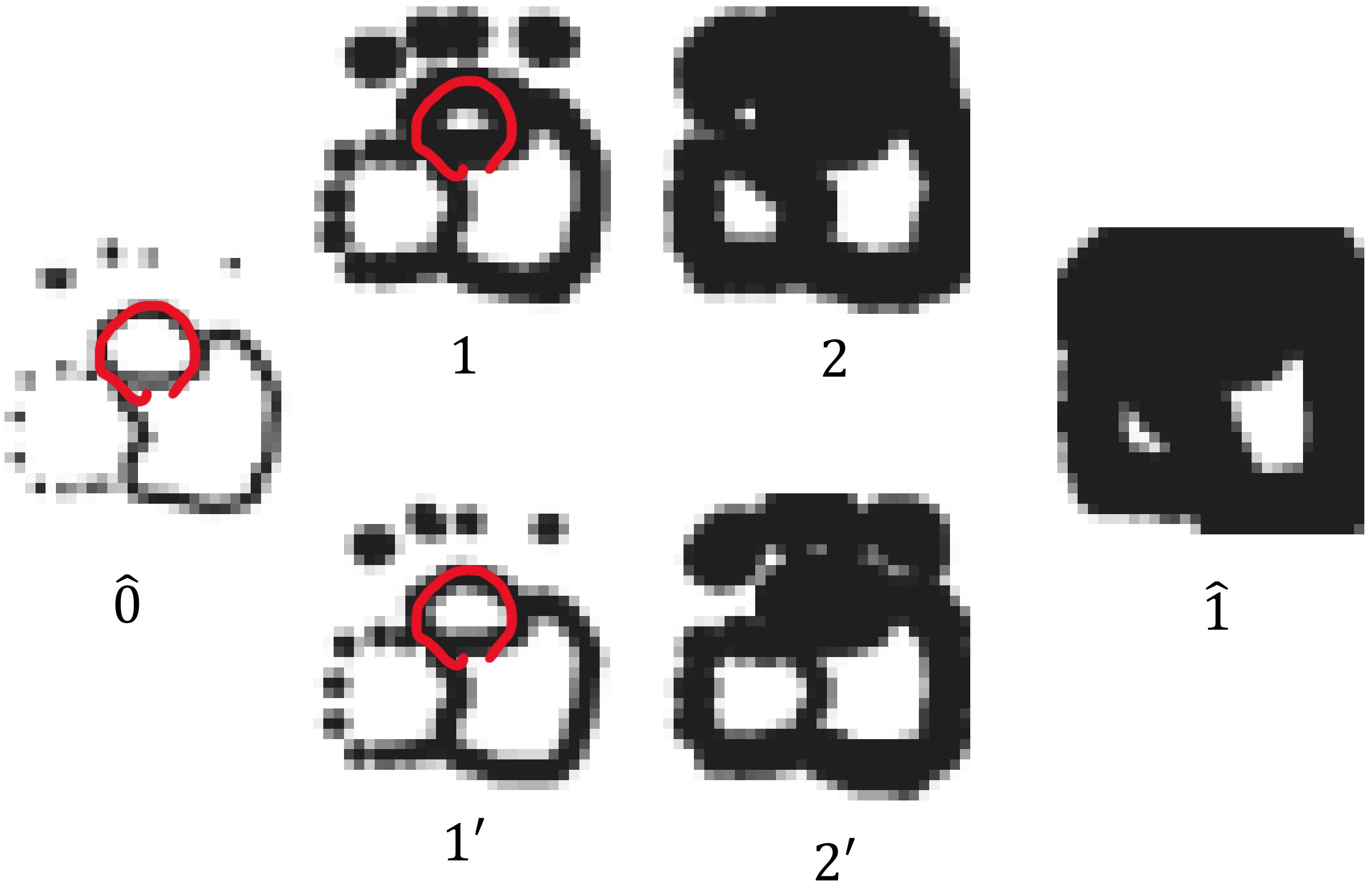
Bipath filtration of image data (30 × 30pixel)

1st.



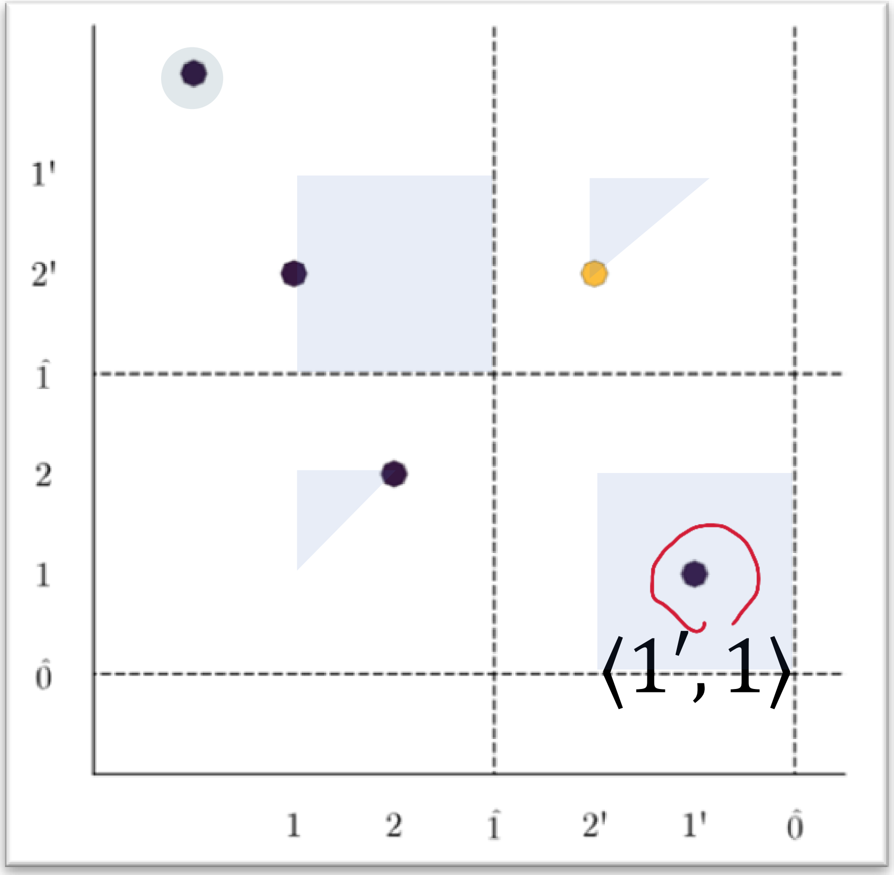
Colors are changed by the multiplicity of intervals.

Examples of bipath PD



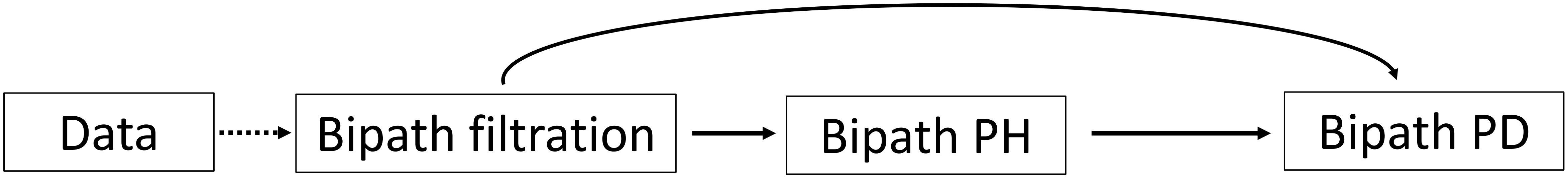
Bipath filtration of image data (30 $\times$ 30pixel)

1st.



Colors are changed by the multiplicity of intervals.

Implementation: Computing Bipath PD



Implementation: Computing bipath PD

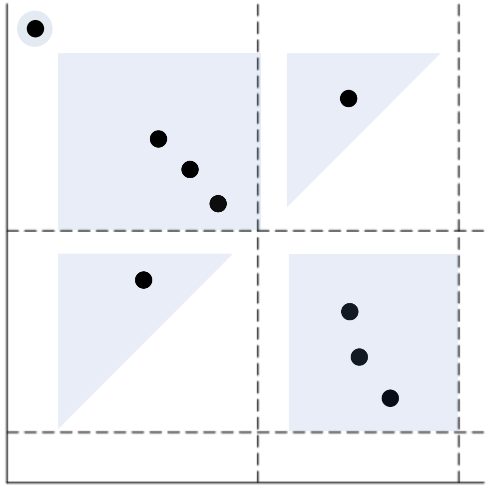
We gave a software for computing bipath PD on GitHub.
(https://github.com/ShunsukeTada1357/Bipathposets)

Input

$$\begin{tikzcd}[column sep=small] & S_1 \arrow[r, hook] & S_2 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[dr, hook] & \\ S_{\hat{0}} \arrow[ur, hook] \arrow[dr, hook] & & & & & S_{\hat{1}} \\ & S_{1'} \arrow[r, hook] & S_{2'} \arrow[r, hook] & \cdots \arrow[r, hook] & S_{m'} \arrow[ur, hook] & \end{tikzcd}$$

Bipath filtration
(of simplicial complex)

Output



Bipath PD

Remark.

The computational algorithm is given in [Aoki-Escolar-T, Algorithm 2, 25].

An algorithm for bipath PD

Input: S bipath filtration of simplicial complex

$$S:\quad \begin{tikzcd}[column sep=small] & S_1 \arrow[r] & \cdots \arrow[r] & s_n \arrow[dr] & \\ S_{\hat{0}} \arrow[ur] \arrow[dr] & & & & S_{\hat{1}} \\ & S_{1'} \arrow[r] & \cdots \arrow[r] & s_{m'} \arrow[ur] & \end{tikzcd}$$

<u>Step 0</u>. Separate a bipath filtration S into S_U and S_D.

$$\begin{array}{lcccccccccc} S_U: & S_{\hat{0}} & \to & S_1 & \to & \cdots & \to & s_n & \to & S_{\hat{1}} \\ & \parallel & & & & & & & & \parallel \\ S_D: & S_{\hat{0}} & \to & S_{1'} & \to & \cdots & \to & s_{m'} & \to & S_{\hat{1}} \end{array}$$

An algorithm for bipath PD

Input: S bipath filtration of simplicial complex

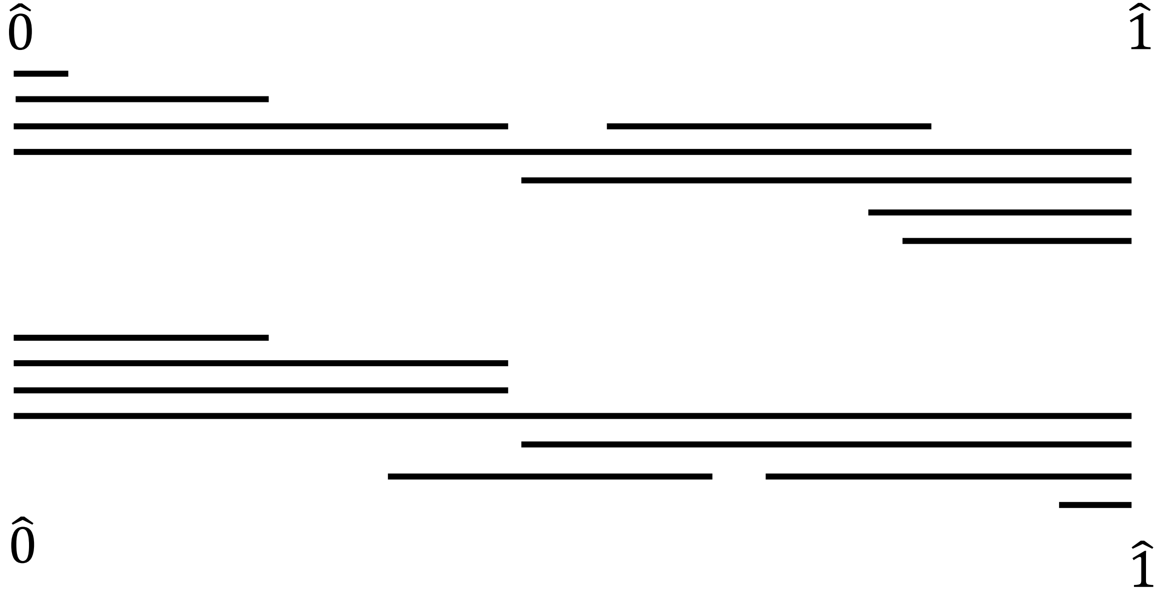
$$S\colon \quad S_{\hat{0}} \begin{array}{l} \nearrow \; S_1 \to \cdots \to S_n \; \searrow \\ \searrow \; S_{1'} \to \cdots \to S_{m'} \; \nearrow \end{array} S_{\hat{1}}$$

Step 1. Get intervals of S_U and S_D by standard algorithm.

$$\begin{array}{l} S_U\colon\; S_{\hat{0}} \to S_1 \to \cdots \to S_n \to S_{\hat{1}} \\ \qquad\;\; \parallel \qquad\qquad\qquad\qquad\qquad\quad \parallel \\ S_D\colon\; S_{\hat{0}} \to S_{1'} \to \cdots \to S_{m'} \to S_{\hat{1}} \end{array}$$

$H_q(S_U) \cong X =$

$H_q(S_D) \cong Y =$



An algorithm for bipath PD

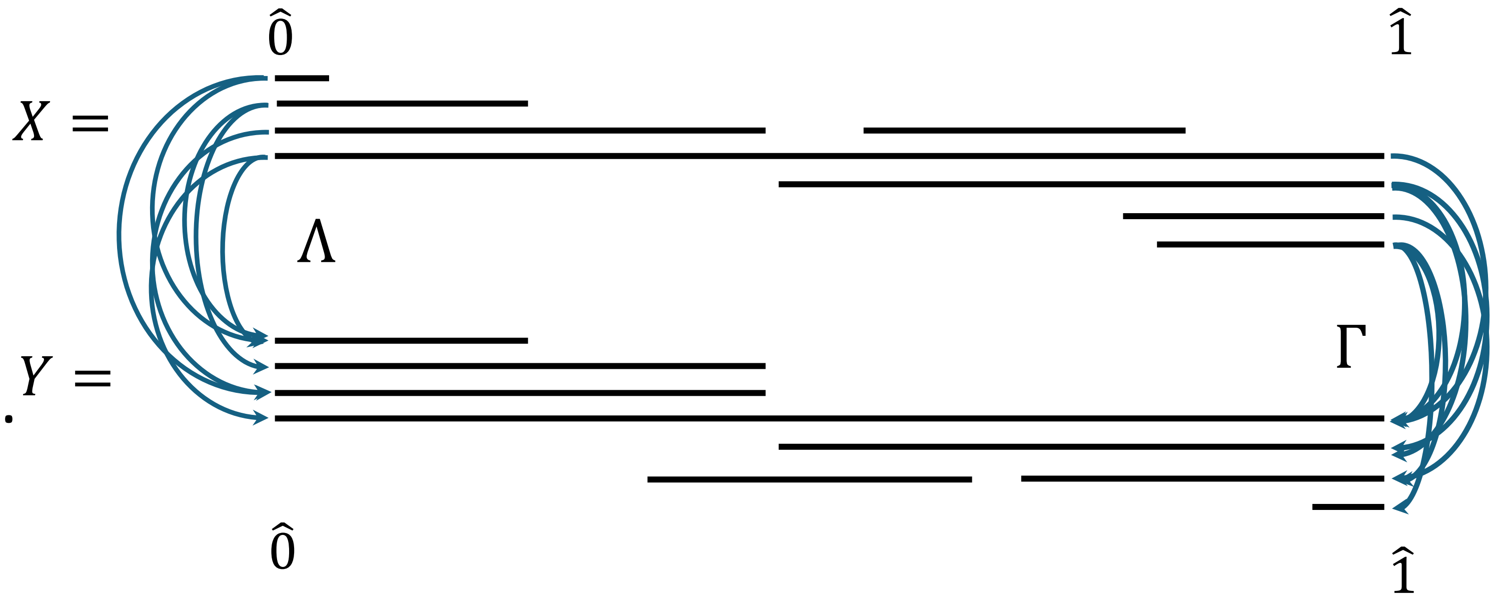
Input: S bipath filtration of simplicial complex

$$
\begin{tikzcd}[row sep=small]
 & S_1 \arrow[r] & \cdots \arrow[r] & s_n \arrow[dr] & \\
S: \quad S_{\hat{0}} \arrow[ur] \arrow[dr] & & & & S_{\hat{1}} \\
 & S_{1'} \arrow[r] & \cdots \arrow[r] & s_{m'} \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{array}{l}
S_U:\ S_{\hat{0}} \to S_1 \to \cdots \to s_n \to S_{\hat{1}} \\
\qquad\ \ \shortparallel \qquad\qquad\qquad\qquad\qquad\quad \shortparallel \\
S_D:\ S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to S_{\hat{1}}
\end{array}
$$

<u>Step 2</u>. Compute change-of-basis matrices

$$\Lambda: X_{\hat{0}} \to Y_{\hat{0}}$$
$$\Gamma: X_{\hat{1}} \to Y_{\hat{1}}.$$



- The size of matrices Λ and Γ is smaller than #intervals.

-> Usually smaller than #simplices.

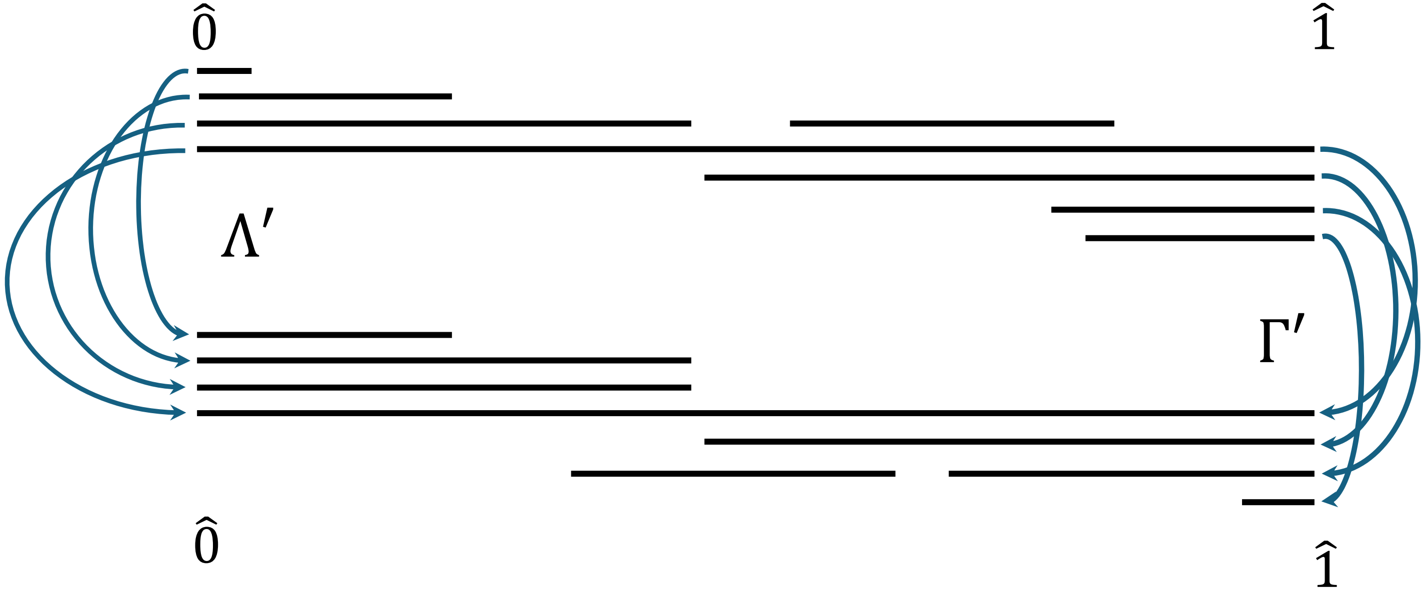
An algorithm for bipath PD

Input: S bipath filtration of simplicial complex

$$
\begin{tikzcd}[column sep=small]
 & & S_1 \arrow[r] & \cdots \arrow[r] & s_n \arrow[dr] & \\
S: & S_{\hat{0}} \arrow[ur] \arrow[dr] & & & & S_{\hat{1}} \\
 & & S_{1'} \arrow[r] & \cdots \arrow[r] & s_{m'} \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
S_U: & S_{\hat{0}} \arrow[r] \arrow[d, equal] & S_1 \arrow[r] & \cdots \arrow[r] & s_n \arrow[r] & S_{\hat{1}} \arrow[d, equal] \\
S_D: & S_{\hat{0}} \arrow[r] & S_{1'} \arrow[r] & \cdots \arrow[r] & s_{m'} \arrow[r] & S_{\hat{1}}
\end{tikzcd}
$$

Step 3. Reduce Λ and Γ to permutation matrices with preserving upper and lower interval decompositions.



An algorithm for bipath PD

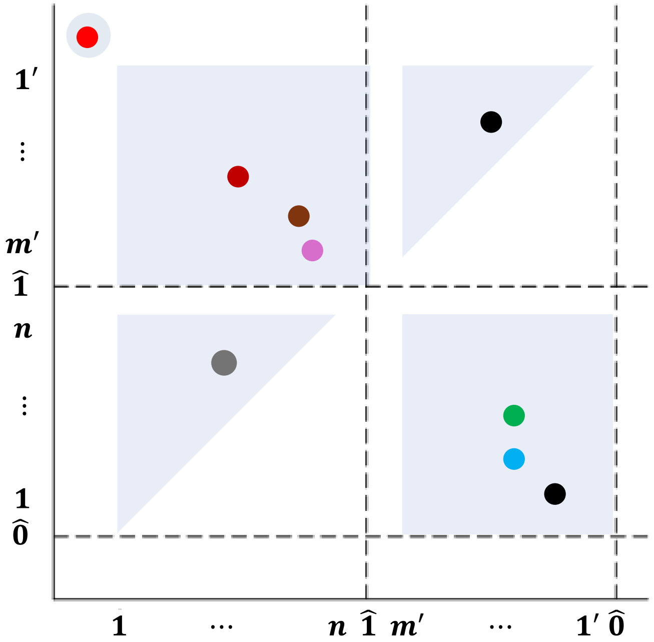
Input: S bipath filtration of simplicial complex

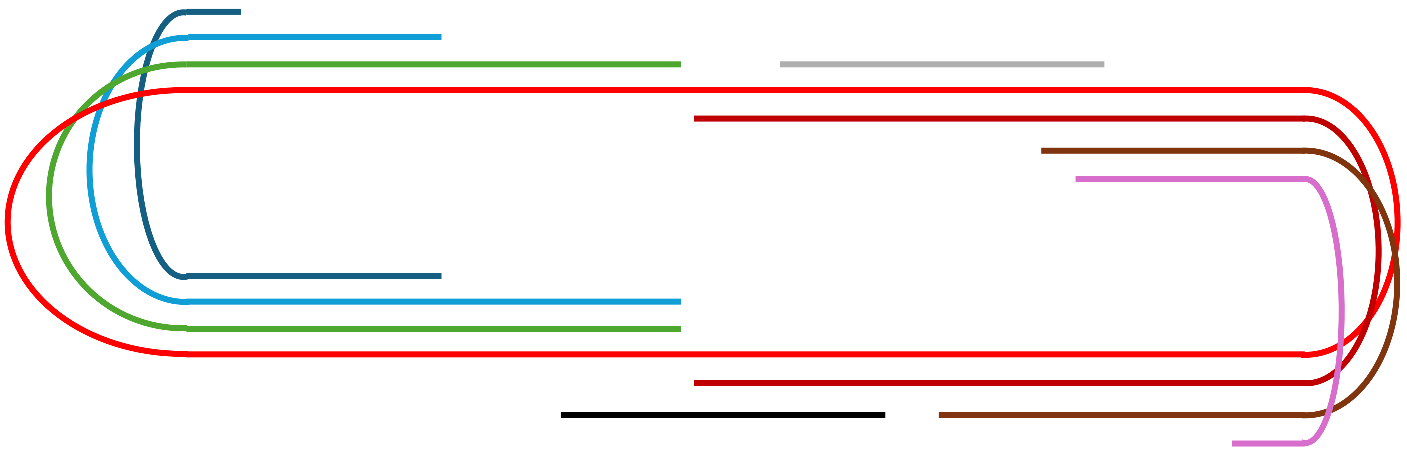
$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & S_1 \arrow[r] & \cdots \arrow[r] & s_n \arrow[dr] & \\
S: & S_{\hat{0}} \arrow[ur] \arrow[dr] & & & & S_{\hat{1}} \\
 & & S_{1'} \arrow[r] & \cdots \arrow[r] & s_{m'} \arrow[ur] &
\end{tikzcd}
$$

Step 4. Connect upper and lower intervals, and get intervals.

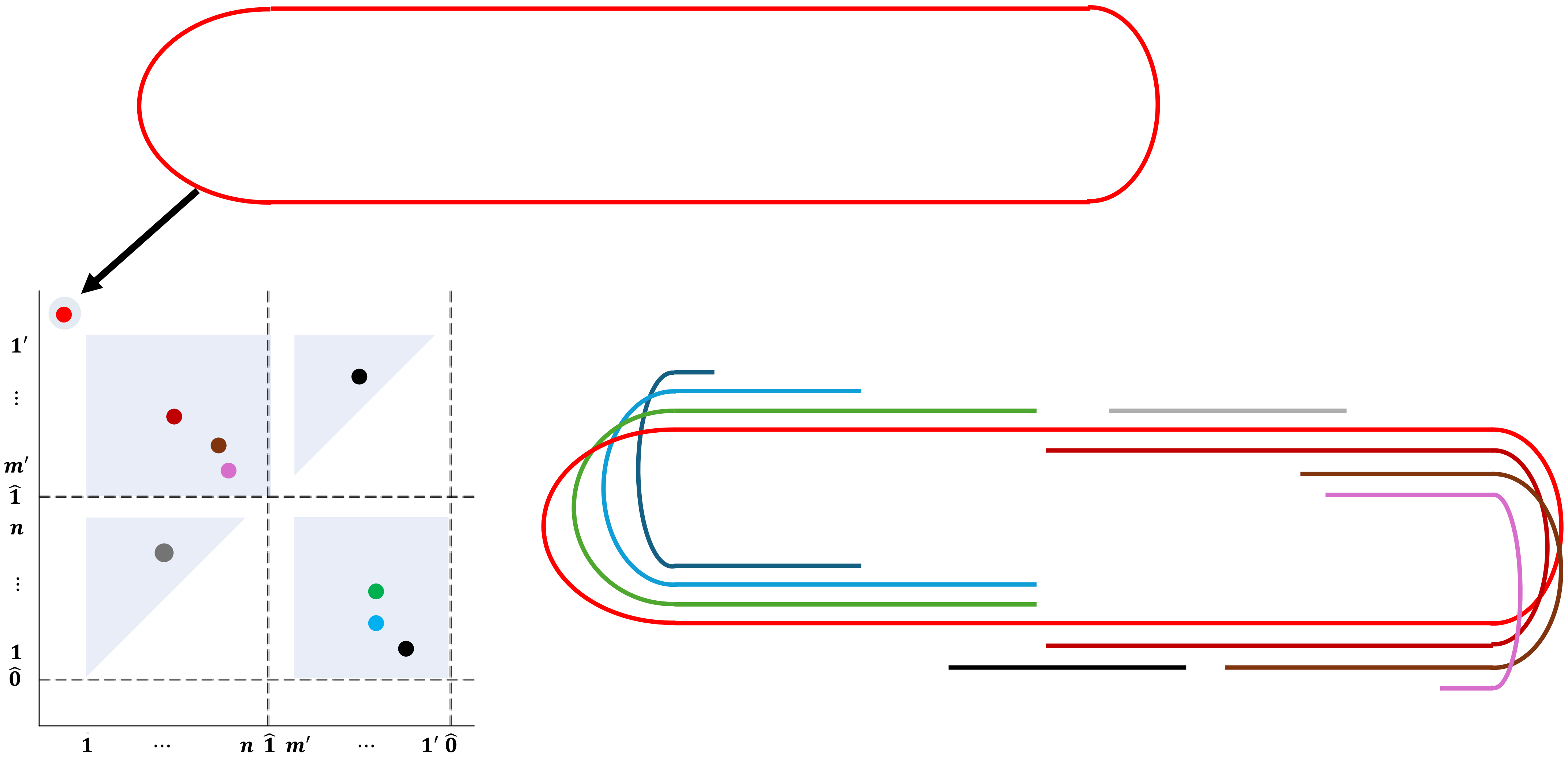
$$
\begin{array}{llllll}
S_U: & S_{\hat{0}} \to S_1 \to \cdots \to s_n \to & S_{\hat{1}} \\
 & \parallel & \parallel \\
S_D: & S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to & S_{\hat{1}}
\end{array}
$$

Output:



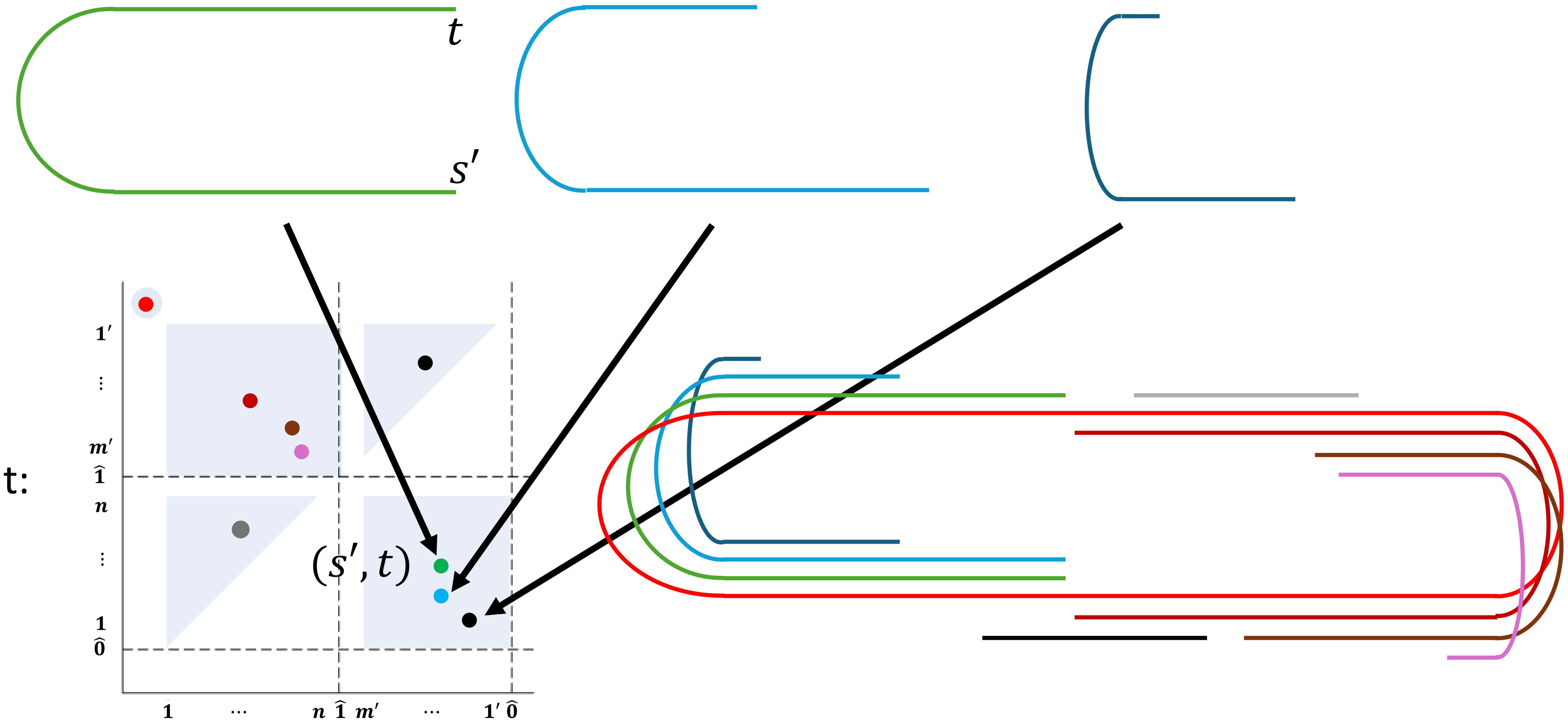


An algorithm for bipath PD



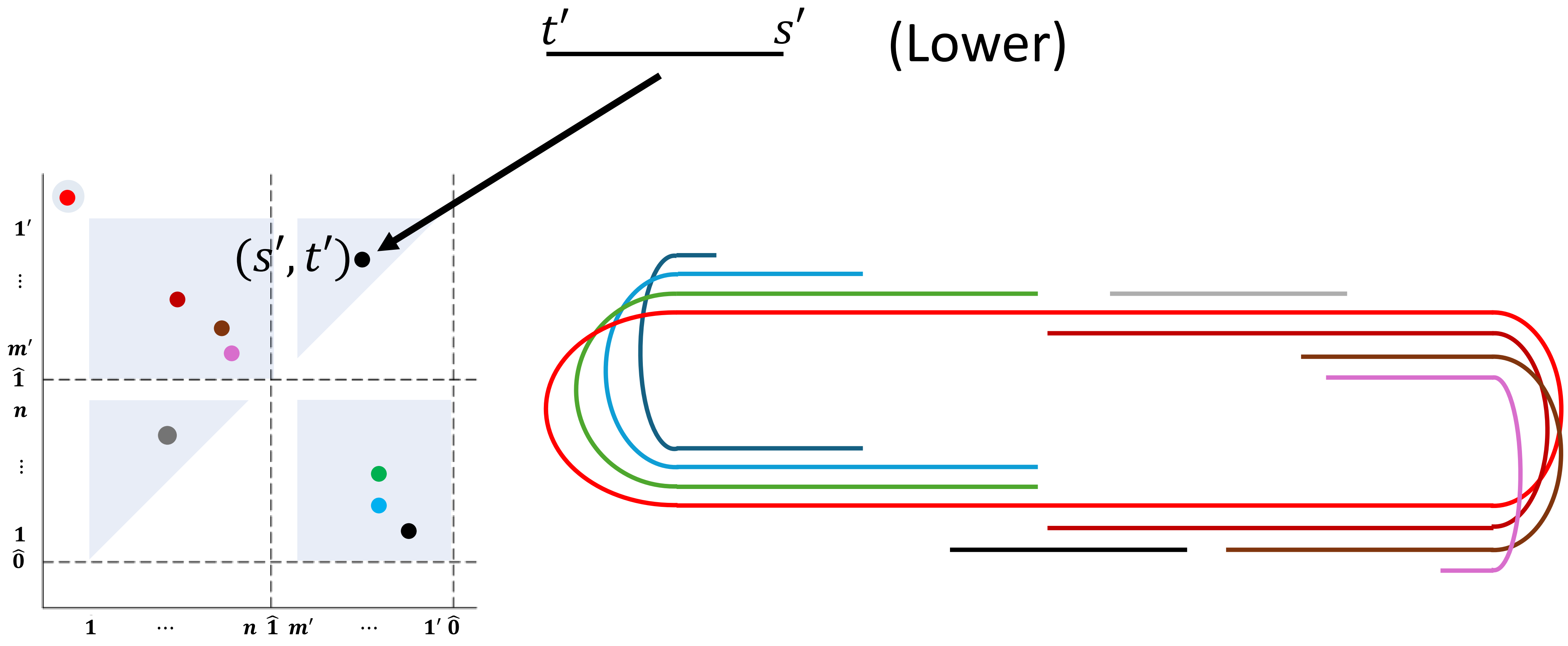
An algorithm for bipath PD

Output:



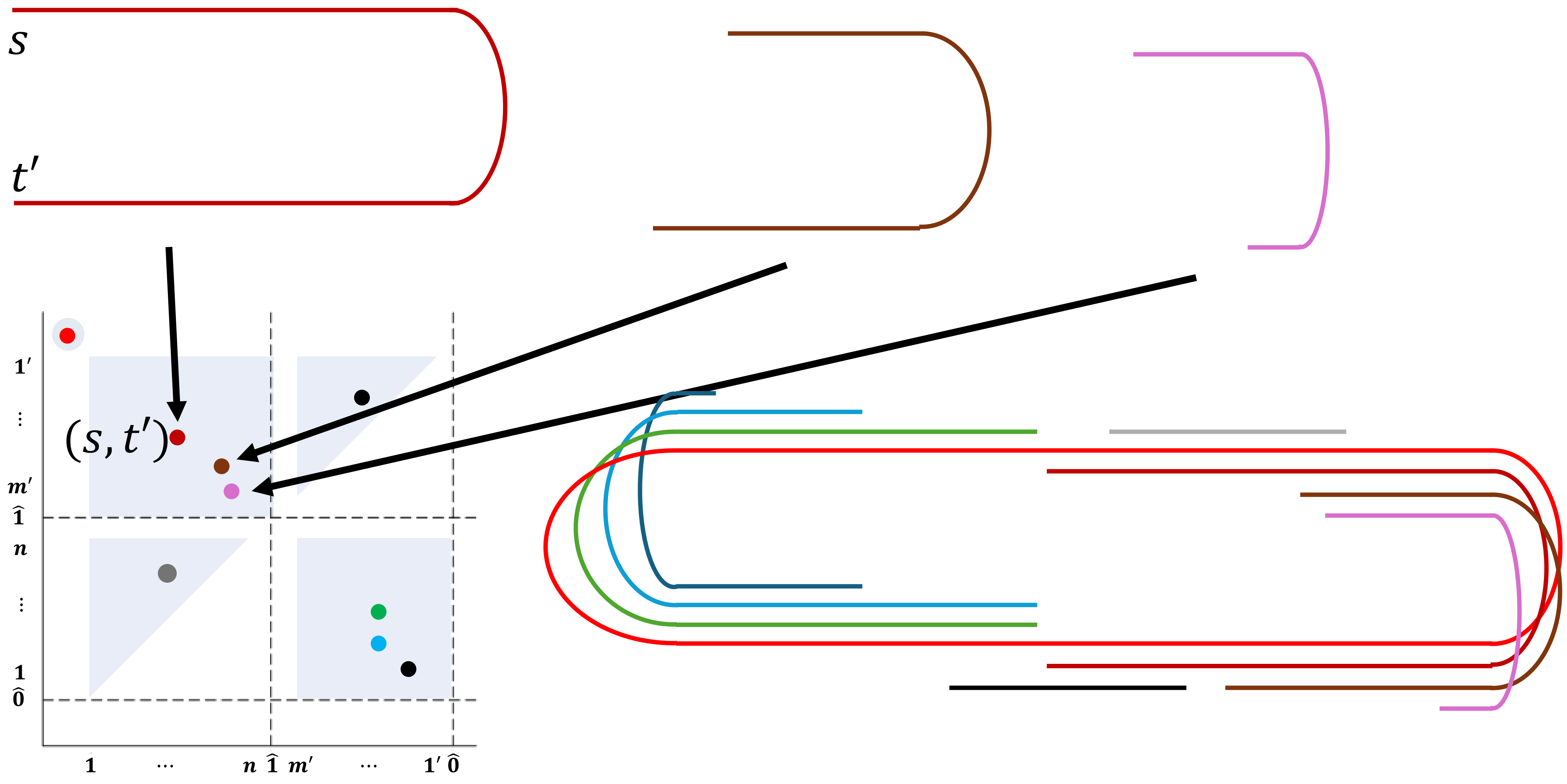
An algorithm for bipath PD

Output:



An algorithm for bipath PD

Output:



An algorithm for bipath PD

Note.

・ We get a bipath PD by 2 times of standard algorithm for PH and matrix operations on Λ and Γ (whose size depend on intervals).

->Bipath PD can be computed without much more effort than standard algorithm.

・ Mathematical foundation for our algorithm is in our paper “Bipath Persistence”.

“Stability properties”

Background: Stability theorem for standard PH

Stability theorem (see [Frédéric Chazal, et al. '09] for example)

Let f and g be real-valued functions on a top. sp. X. Then, we have

$$d_{\mathrm{B}}\left(\mathcal{B}(V_f), \mathcal{B}(V_g)\right) \leq ||f - g||.$$

Data ⟶ Filtration ⟶ PH ⟶ PD

$f: X \to \mathbb{R}$

$(f \leq r)_{r \in \mathbb{R}}$

$(f \leq r) \coloneqq \{x \in X \mid f(x) \leq r\}$

V_f

$\mathcal{B}(V_f)$

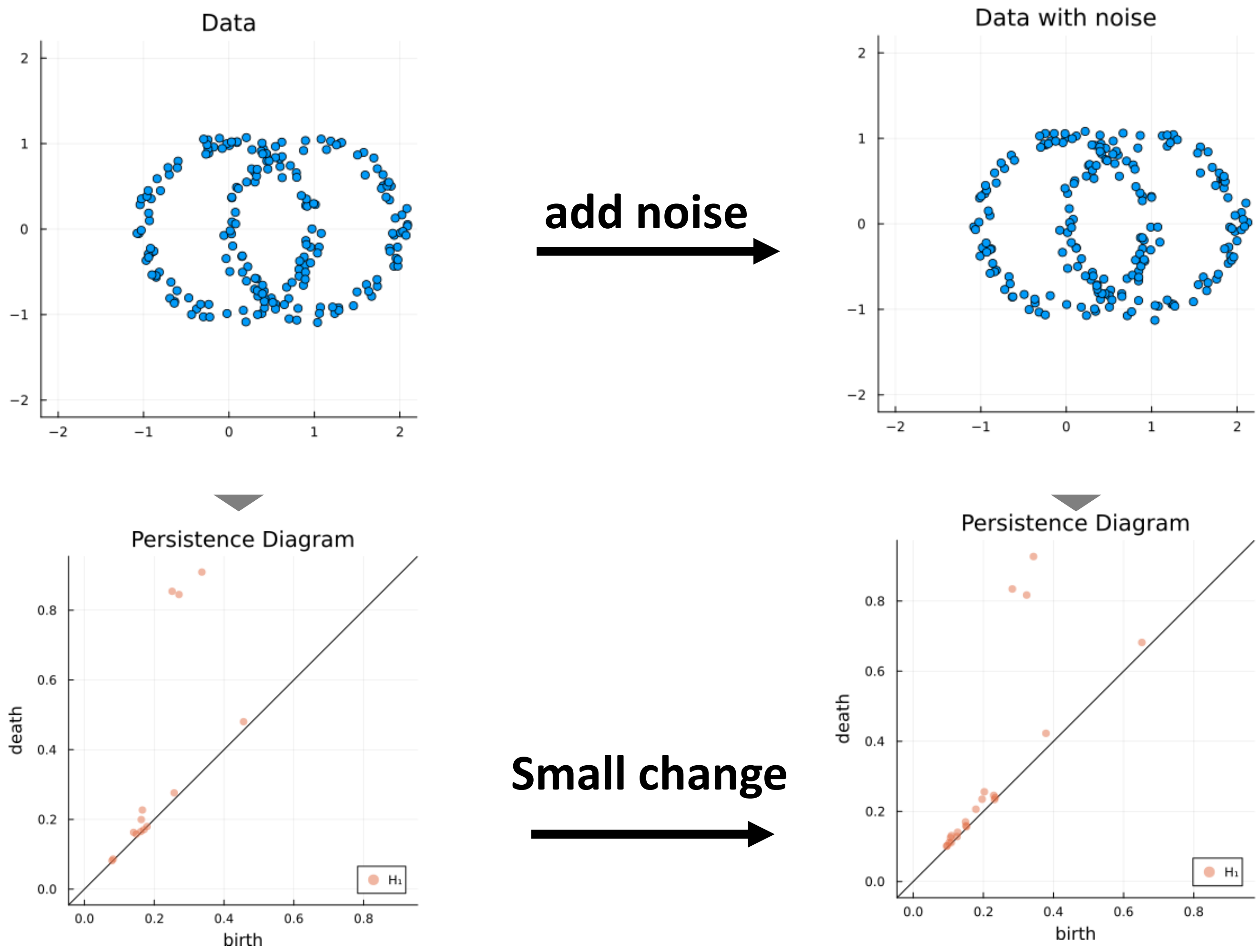
$\rightsquigarrow$ Small changes in data imply small changes in the PD.

It justifies the use of PH for studying noisy data.

・David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. Discrete& Computational Geometry, 37:103–120, 2007.
・Frédéric Chazal, David Cohen-Steiner, Marc Glisse, Leonidas J Guibas, and Steve Y Oudot. Proximityof persistence modules and their diagrams. In Proceedings of the twenty-fifth annual symposium onComputational geometry, pages 237–246, 2009.

Background: Stability theorem for standard PH

Example



Background: Stability theorem for standard PH

Recall that stability theorem can be deduced by the isometry theorem.

Isometry theorem [Lesnick '15]

Let V and W be $\mathbb{R}$-persistence modules. Then, V and W are ϵ-interleaved if and only if there exist an ϵ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. Thus, we have

$$d_{\mathrm{B}}\big(\mathcal{B}(V), \mathcal{B}(W)\big) = d_{\mathrm{I}}(V, W).$$

Michael Lesnick. The theory of the interleaving distance on multidimensional persistence modules. Found. Comput. Math. 2015.

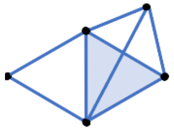
Stability theorem for bipath PD

Theorem [T, '25, Theorem 4.1](Stability theorem for bipath PD).

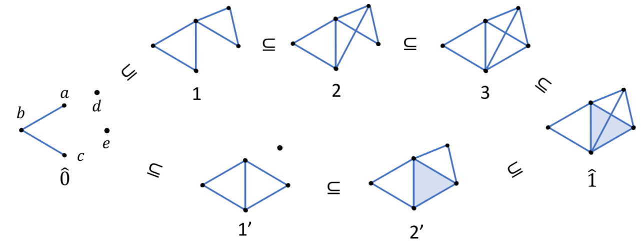
Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be bipath functions on top. sp. X satisfying (◆). Then, we have the following inequality:

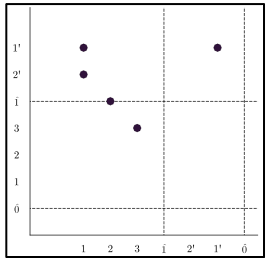
$$d_{\mathrm{B}}\big(\mathcal{B}(V_f), \mathcal{B}(V_g)\big) \leq ||f, g||_B.$$

$$\begin{array}{ccccccc} \boxed{\text{Data}} & \longrightarrow & \boxed{\text{Bipath filtration}} & \longrightarrow & \boxed{\text{Bipath PH}} & \longrightarrow & \boxed{\text{Bipath PD}} \\ f & & (f \leq b)_{b \in B} & & V_f & & \mathcal{B}(V_f) \end{array}$$

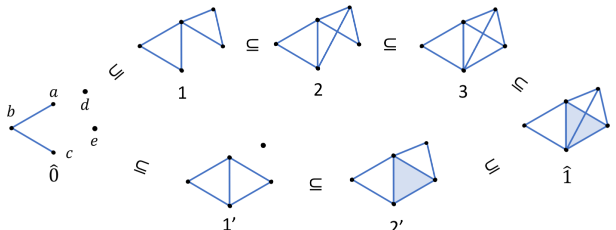
$X :=$ 

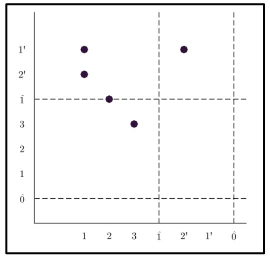
$f = (f_1, f_2)$



$\mathcal{B}(V_f)$ 

$g = (g_1, g_2)$



$\mathcal{B}(V_g)$ 

Stability theorem for bipath PD

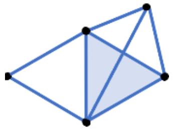
Theorem [T, ’25, Theorem 4.1](Stability theorem for bipath PD).

Let $f=(f_1,f_2)$ and $g=(g_1,g_2)$ be bipath functions on top. sp. X satisfying (◆). Then, we have the following inequality:

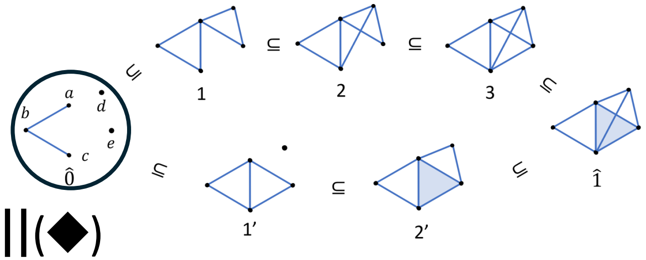
$$d_{\mathrm{B}}\big(\mathcal{B}(V_f),\mathcal{B}(V_g)\big)\leq ||f,g||_B.$$

Data $\longrightarrow$ Bipath filtration $\longrightarrow$ Bipath PH $\longrightarrow$ Bipath PD

f $\quad (f\leq b)_{b\in B}$ $\quad V_f$ $\quad \mathcal{B}(V_f)$

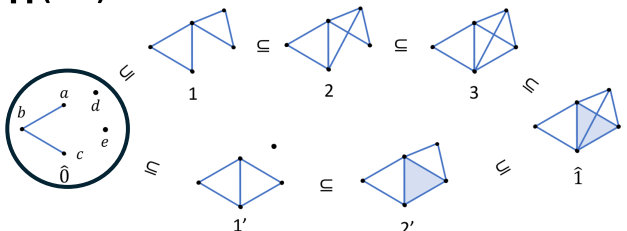
$X :=$ 

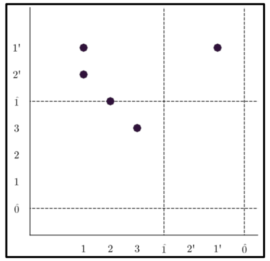
$f=(f_1,f_2)$

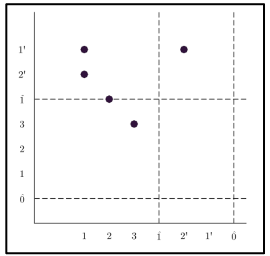


||(◆)

$g=(g_1,g_2)$



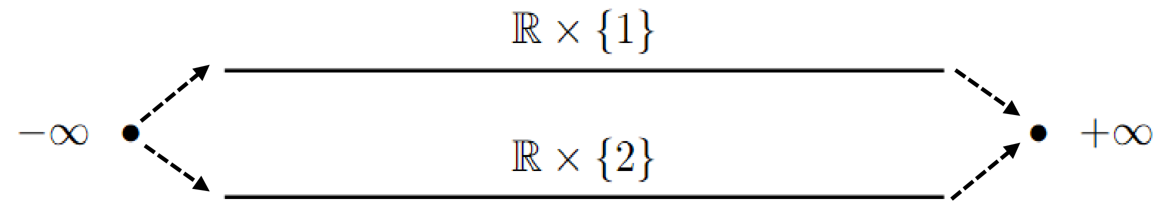
$\mathcal{B}(V_f)$ 

$\mathcal{B}(V_g)$ 

Stability theorem for bipath PD

To discuss stability, we consider continuous bipath poset B.

$$B_{n,m}\colon \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ -\infty \arrow[ur] \arrow[dr] & & & & & +\infty \\ & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd} \quad \rightsquigarrow \quad B\colon$$



$$(\mathbb{R} \times \{1\}) \sqcup (\mathbb{R} \times \{2\}) \sqcup \{-\infty, +\infty\}$$

Isometry theorem for bipath persistence [T '25]

Let V and W be B-persistence modules. Then V and W are ϵ-interleaved if and only if there exist an ϵ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. Thus, we have $d_{\mathrm{B}}(\mathcal{B}(V), \mathcal{B}(W)) = d_{\mathrm{I}}(V, W)$.

$\rightsquigarrow$
- Setting for the definitions of d_{I} and d_{B}.
- Graph-theoretic approach in the general setting.
- Return to the bipath setting.

Stability theorem: Setting

- Let k be a field, and let P be a poset.
- A P-persistence module is an object in $\mathrm{rep}_k(P) \coloneqq \mathrm{Fun}(P, \mathrm{vect}_k)$.

  Equivalently, a P-persistence module $V = \left(V_p, V(p \leq q)\right)_{p \leq q \in P}$ s. t.

  - V_p is a fin. dim k-vector space.
  - $V(p \leq q)\colon V_p \to V_q$ is a linear map satisfying
    $$V(p \leq r) = V(q \leq r) \circ V(p \leq q) \ (\forall\, p \leq q \leq r \ \in P)$$
  - $V(p \leq p) = \mathrm{id}_P$

- For $V \cong \bigoplus_{\gamma \in \Gamma} V_\gamma \in \mathrm{rep}_k(P)$ (V_γ: indecomposable), set $\mathcal{B}(V)$:={{ $V_\gamma \,|\, \gamma \in \Gamma$}}

Stability theorem: Setting

・A *translation* on P is an order-preserving map $h\colon P \to P$ s. t. $p \leq h(p)$ for every $p \in P$.

・Fix a family of translations $\Lambda := \{\Lambda_\epsilon\}_{\epsilon \in \mathbb{R}_{\geq 0}}$ on P satisfying:

$$\Lambda_0 = \mathrm{id}_P \text{ and } \Lambda_{\epsilon+\zeta} = \Lambda_\epsilon \circ \Lambda_\zeta \text{ for all } \epsilon, \zeta \in \mathbb{R}_{\geq 0}.$$

Interleaving and bottleneck distances are defined w. r. t. $\Lambda := \{\Lambda_\epsilon\}_{\epsilon \in \mathbb{R}_{\geq 0}}$.

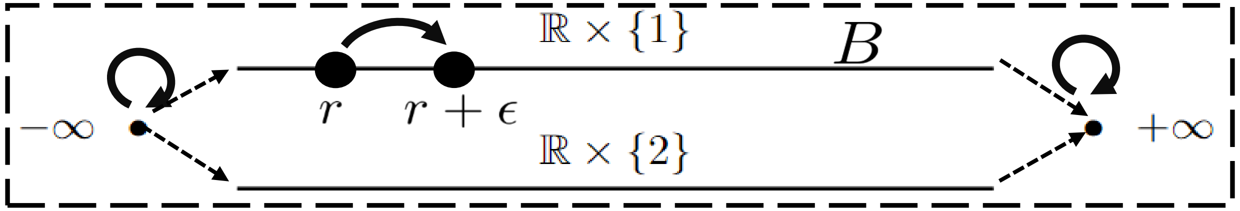
Example

Let $P := \mathbb{R}$. We define $\Lambda^{\mathbb{R}} := \{\Lambda^{\mathbb{R}}_\epsilon\}_{\epsilon \in \mathbb{R}_{\geq 0}}$ by ... $) := r + \epsilon$ for every $r \in \mathbb{R}$.

Example [T, '25, Definition 3.4].

Let B be the bipath poset. We define $\Lambda^B_\epsilon := \{\Lambda^B_\epsilon\}_{\epsilon \in \mathbb{R}_{\geq 0}}$ by

$\Lambda^B_\epsilon(\pm\infty) := \pm\infty$, and $\Lambda^B_\epsilon((r,i)) := (r+\epsilon, i)$ for $(r,i) \in \mathbb{R} \times \{i\}$ $(i = 1,2)$.



Stability theorem: Setting

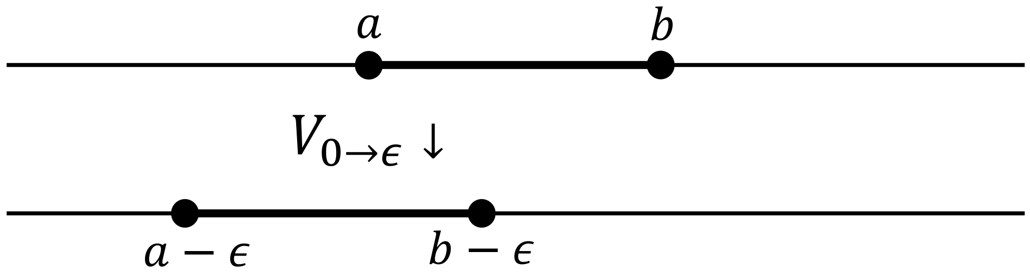
Let V, W be P-persistence modules, and $\epsilon \geq 0$.

・We write $V(\epsilon) \coloneqq V \circ \Lambda_\epsilon \in \mathrm{rep}_k(P)$
(this gives a functor $(\cdot)(\epsilon)\colon \mathrm{rep}_k(P) \to \mathrm{rep}_k(P)$),
then, we have the induced morphism $V_{0\to\epsilon}\colon V \to V(\epsilon)$.

$P = \mathbb{R}, a \leq b \in \mathbb{R}$.

$V := k[a,b]$:

$V(\epsilon) = k[a-\epsilon, b-\epsilon]$:



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then, we have the induced morphism $V_{0\to\epsilon}\colon V \to V(\epsilon)$.

・We say that V and W are ϵ-interleaved and write $V \sim_\epsilon W$ if there is a pair of morphisms $\alpha\colon V \to W(\epsilon)$ and $\beta\colon W \to V(\epsilon)$ s. t.

$$V_{0\to 2\epsilon} = \beta(\epsilon) \circ \alpha \text{ and } W_{0\to 2\epsilon} = \alpha(\epsilon) \circ \beta.$$

$$\begin{tikzcd}
V_p \arrow[r] \arrow[rr, bend left, "V_{0\to 2\epsilon}"] \arrow[dr, "\alpha_p"'] & V(\epsilon)_p \arrow[r] \arrow[dr, "\alpha(\epsilon)_p"] & V(2\epsilon)_p \\
W_p \arrow[r] \arrow[rr, bend right, "W_{0\to 2\epsilon}"'] \arrow[ur, "\beta_p"'] & W(\epsilon)_p \arrow[r] \arrow[ur, "\beta(\epsilon)_p"'] & W(2\epsilon)_p
\end{tikzcd}$$

$\forall p \in P$

Stability theorem: Setting

Let V, W be P-persistence modules, and $\epsilon \geq 0$.

・We write $V(\epsilon) \coloneqq V \circ \Lambda_\epsilon \in \mathrm{rep}_k(P)$
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$$V_{0\to 2\epsilon} = \beta(\epsilon) \circ \alpha \text{ and } W_{0\to 2\epsilon} = \alpha(\epsilon) \circ \beta.$$

Definition (Interleaving distance)

The interleaving distance between P-persistence modules V and W is defined by $d_{\mathrm{I}}^{\Lambda}(V, W) \coloneqq \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid V \sim_\epsilon W\}$.

Stability theorem: Setting

We say that a P-persistence module V is ϵ-trivial if $V_{0\to\epsilon} = 0$.

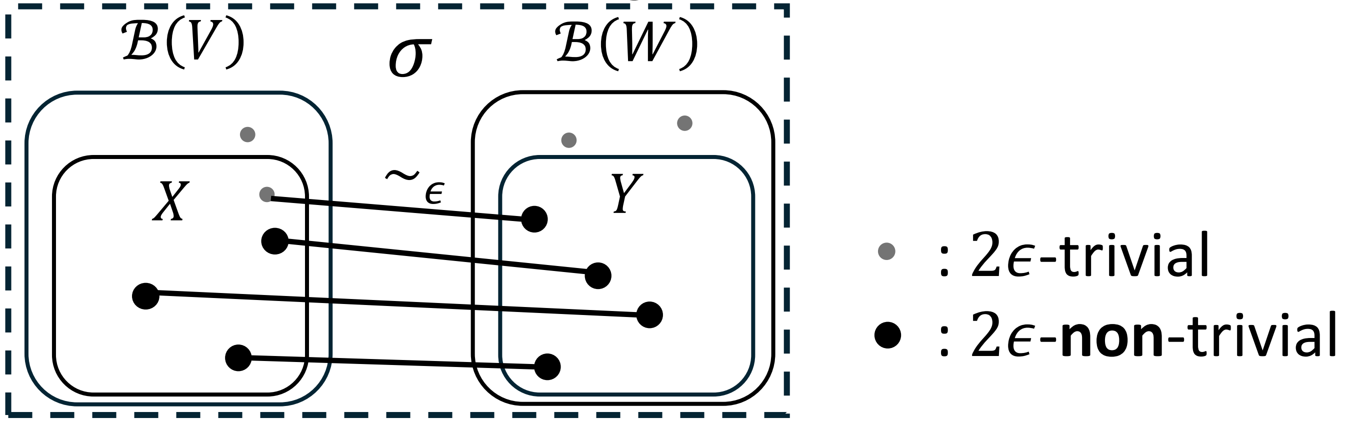
Definition (ϵ-matching)

Let V and W be P-persistence modules. An ϵ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$ is a partial matching $\sigma: \mathcal{B}(V) \supseteq X \overset{1:1}{\to} Y \subseteq \mathcal{B}(W)$ satisfying:

- Every $I \in (\mathcal{B}(V) \sqcup \mathcal{B}(W)) \setminus (X \sqcup Y)$ is 2ϵ-trivial.
- If $\sigma(I) = J$, then $I \sim_\epsilon J$.

We say that V and W are ϵ-matched if there is an ϵ-matching.

Picture of an ϵ-matching σ



Stability theorem: Setting

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- Every $I \in (\mathcal{B}(V) \sqcup \mathcal{B}(W)) \setminus (X \sqcup Y)$ is 2ϵ-trivial.
- If $\sigma(I) = J$, then $I \sim_\epsilon J$.

We say that V and W are ϵ-matched if there is an ϵ-matching.

Definition (Bottleneck distance)

The bottleneck distance between P-persistence modules V and W is defined by $d_{\mathrm{B}}^{\Lambda}(\mathcal{B}(V), \mathcal{B}(W)) := \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid V \text{ and } W \text{ are } \epsilon\text{-matched}\}$.

Stability theorem: Outline.

Remark

Let V and W be P-persistence modules. If V and W are ϵ-matched, then they are ϵ-interleaved. Thus, we have

$$d_{\mathrm{B}}^{\Lambda}(\mathcal{B}(V), \mathcal{B}(W)) \geq d_{\mathrm{I}}^{\Lambda}(V, W).$$

($\because$ An ϵ-matching induces an ϵ-interleaving.)

$\rightsquigarrow$ We observe the converse for the isometry theorem:

V and W are ϵ-interleaved. $\Rightarrow$ V and W are ϵ-matched.

Step1: Interpreting an ϵ-matching as a matching in a bipartite graph

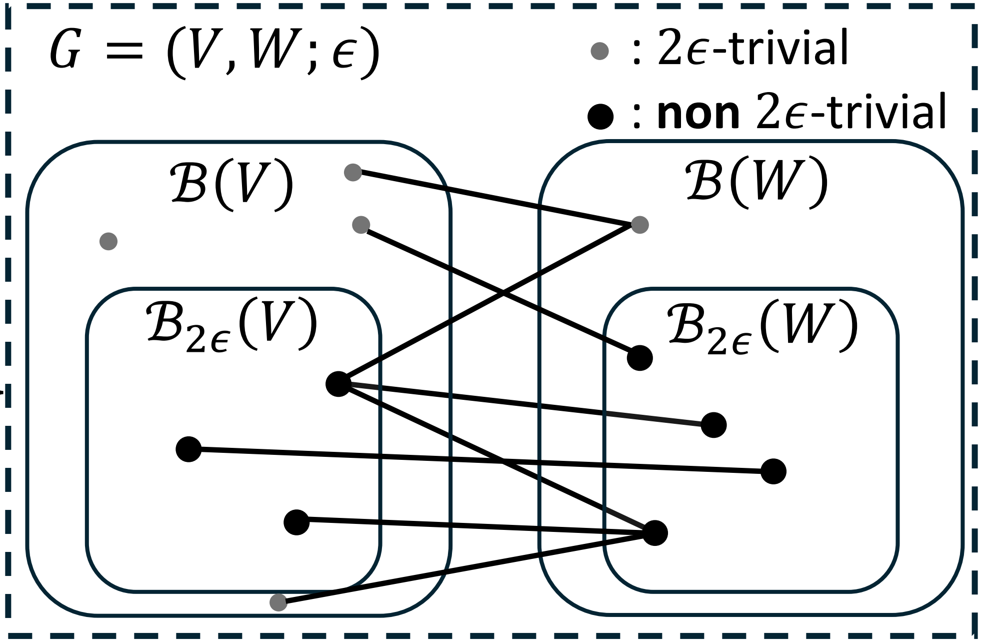
Step2: A sufficient condition for an ϵ-matching using a bipartite graph

Step3: Hall's marriage theorem is useful for showing the sufficient condition.

Step4: In the bipath setting, Step 2 is proved through Step 3.

Stability theorem: Outline Step 1

- Let V, W be P-persistence modules.
- Make a bipartite graph $G = (V, W; \epsilon)$.
  - Vertices $\mathcal{B}(V) \sqcup \mathcal{B}(W)$
  - Edges $\{\{I, J\} \mid I \in \mathcal{B}(V), J \in \mathcal{B}(W), I \sim_\epsilon J\}$
- $\mathcal{B}_{2\epsilon}(V) := \{I \in \mathcal{B}(V) \mid I \text{ is } \mathbf{not}\ 2\epsilon\text{-trivial}\}$



Proposition (1)[Bjerkevik '21, p.4]

The following are equivalent.

(a) V and W are ϵ-matched.

(b) $\exists$ a matching in $G = (V, W; \epsilon)$ that covers $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$.

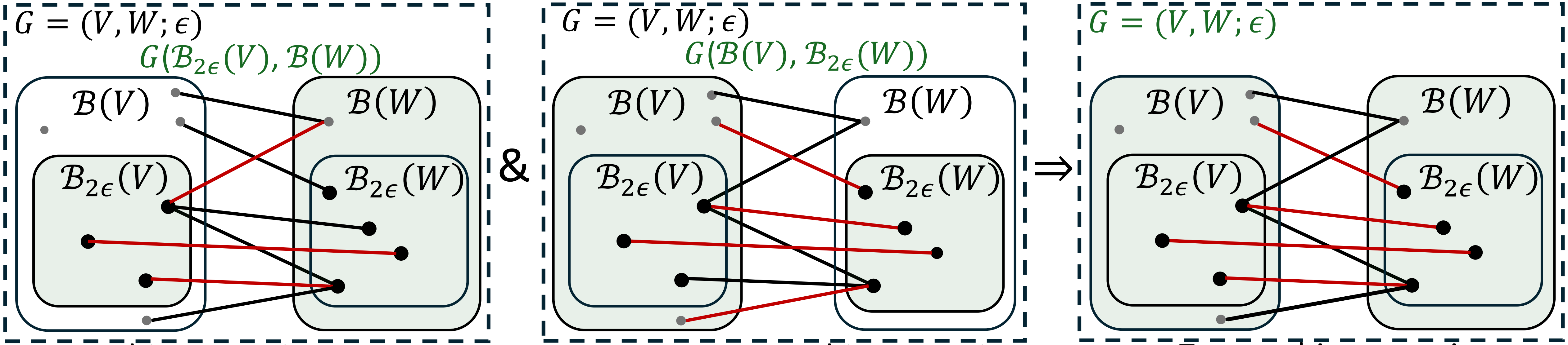
Håvard Bakke Bjerkevik. On the stability of interval decomposable persistence modules. Discrete &Computational Geometry, 66:92–121, 2021

Stability theorem: Outline Step 2

Proposition (2) [cf. Bjerkevik, ’21, p. 111]

Let V and W be P-persistence modules. If the following are satisfied, then there is a matching in $G = (V, W; \epsilon)$ that covers $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$:

- $\exists$ a matching in the full subgraph $G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$that covers $\mathcal{B}_{2\epsilon}(V)$.
- $\exists$ a matching in the full subgraph $G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ that covers $\mathcal{B}_{2\epsilon}(W)$.



$\exists$a matching covering $\mathcal{B}_{2\epsilon}(V)$

$\exists$a matching covering $\mathcal{B}_{2\epsilon}(W)$

$\exists$a matching covering $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$

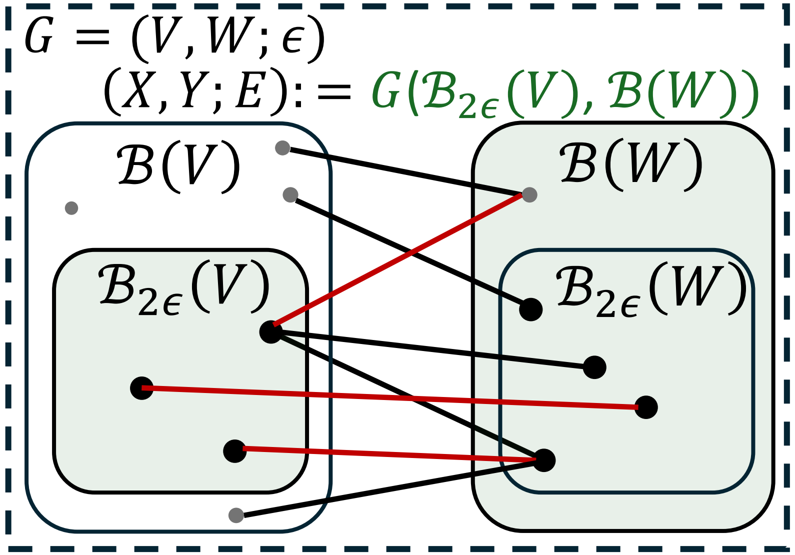
Stability theorem: Outline Step 3

Theorem (3) [Hall, 1935, Theorem 1]

Let $G = (X, Y; E)$ be a bipartite graph such that each vertex $x \in X$ has a finite neighborhood $N_G(x) \subseteq Y$. Then the following are equivalent:

(a) $\exists$ a matching in G that covers X.

(b) For every finite subset $X' \subseteq X$, we have $|X'| \leq |\cup_{x \in X'} N_G(x)|$.



$\exists$a matching covering $\mathcal{B}_{2\epsilon}(V)$

<- Since $V \in \mathrm{rep}_k(P)$ is pointwise finite dimensional, $N_G(x) < \infty$ for every $x \in \mathcal{B}_{2\epsilon}(V)$[Bje21, p.110].

$\rightsquigarrow (X, Y; E) := G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$ satisfies the assumption of Hall's theorem.

$\rightsquigarrow$ Existence of a matching covering $\mathcal{B}_{2\epsilon}(V)$ is equivalent to (b):

$$\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(V), \text{ we have } |X'| \leq |\cup_{x \in X'} N_G(x)|.$$

Philip Hall. On representatives of subsets. Journal of the London Mathematical Society, s1-10(1):26–30,1935.

Stability theorem: Outline Step 3

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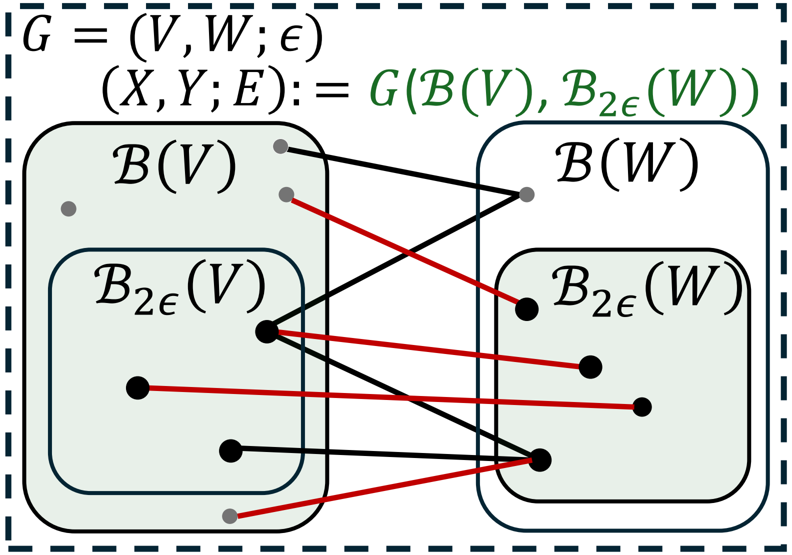


$\exists$a matching covering $\mathcal{B}_{2\epsilon}(W)$

<- Since $W \in \mathrm{rep}_k(P)$ is pointwise finite dimensional, $N_G(x) < \infty$ for every $x \in \mathcal{B}_{2\epsilon}(W)$[Bje21, p.110].

$\rightsquigarrow (X, Y; E) := G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ satisfies the assumption of Hall's theorem.

$\rightsquigarrow$ Existence of a matching covering $\mathcal{B}_{2\epsilon}(W)$ is equivalent to (b):

$$\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(W), \text{ we have } |X'| \leq |\cup_{x \in X'} N_G(x)|.$$

Philip Hall. On representatives of subsets. Journal of the London Mathematical Society, s1-10(1):26–30,1935.

Stability theorem: Outline Step 1, 2, and 3

Let V and W be P-persistence modules.

$\boxed{V \text{ and } W \text{ are } \epsilon\text{-interleaved.}}$

Cf. [Bje, '21, Ex. 5.3]

$\not\Rightarrow$

- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(V)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.
- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(W)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.

$\overset{\text{Prop. (3)}}{\underset{\text{Hall}}{\Longleftrightarrow}}$

- $\exists$a matching in $G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$ that covers $\mathcal{B}_{2\epsilon}(V)$.
- $\exists$a matching in $G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ that covers $\mathcal{B}_{2\epsilon}(W)$.

$\overset{\text{Prop. (2)}}{\Longrightarrow}$

$\boxed{\exists \text{a matching in } G = (V, W; \epsilon) \text{ that covers } \mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W).}$

$\overset{\text{Prop. (1)}}{\Longleftrightarrow}$

$\boxed{V \text{ and } W \text{ are } \epsilon\text{-matched.}}$

Stability theorem: Outline Step 4

Let V and W be $\boldsymbol{B}$**-persistence modules**.

・ B-persistence modules are interval-decomposable, with each interval determined by two elements of B.
・ Λ_ϵ^B is a poset isomorphism ($\forall \epsilon \in \mathbb{R}_{\geq 0}$)

$\boxed{V \text{ and } W \text{ are } \epsilon\text{-interleaved}}$

$\overset{[\mathrm{T}, \text{'}25]}{\Longrightarrow}$

- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(V)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.
- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(W)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.

$\overset{\text{Prop. (3)}}{\underset{\text{Hall}}{\Longleftrightarrow}}$

- $\exists$a matching in $G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$ that covers $\mathcal{B}_{2\epsilon}(V)$.
- $\exists$a matching in $G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ that covers $\mathcal{B}_{2\epsilon}(W)$.

$\overset{\text{Prop. (2)}}{\Longrightarrow}$

$\boxed{\exists \text{a matching in } G = (V, W; \epsilon) \text{ that covers } \mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W).}$

$\overset{\text{Prop. (1)}}{\Longleftrightarrow}$

$\boxed{V \text{ and } W \text{ are } \epsilon\text{-matched.}}$ $\rightsquigarrow d_B^{\Lambda_\epsilon^B}(\mathcal{B}(V), \mathcal{B}(W)) = d_I^{\Lambda_\epsilon^B}(V, W)$

Summary

Bipath PH is an extension of standard PH (so it is nice tool!).

Interval decomposability	○
Visualization (Bipath PD)	○
Algorithm (implementation)	○
Stability theorem for bipath PDs	○
Inverse analysis	-
Application	-

Discussion

・Inverse analysis for bipath persistent homology is required.

・Application of bipath PH to real data. ⇝ We recently discussed the use of it for image data analysis with material scientists.

Thank you for your listening.

補助

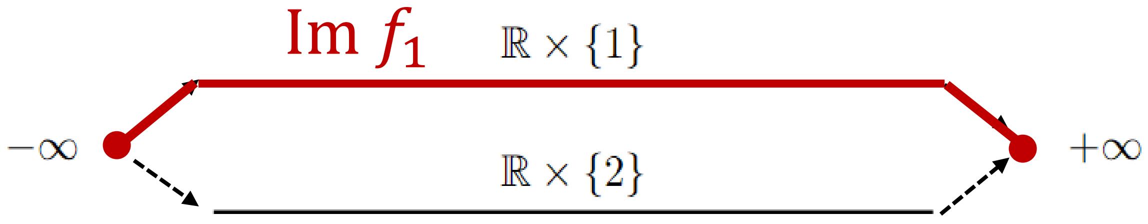
Stability theorem for bipath PD: Bipath sublevelset filtration

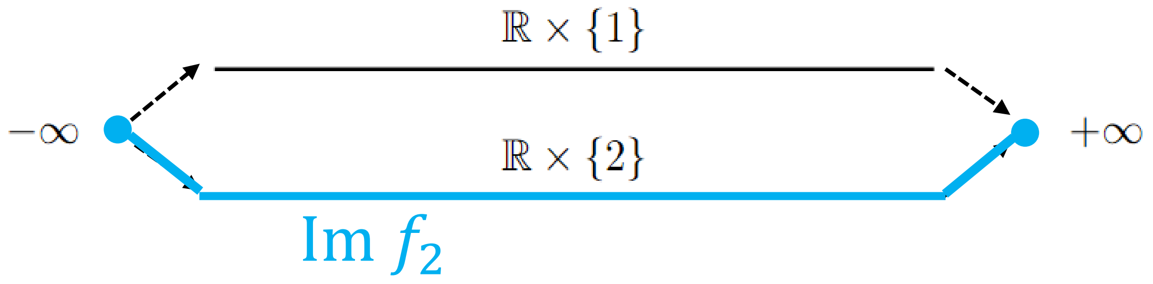
Definition (Bipath function)

A *bipath function* f on top. sp. X is a pair of B-valued functions f_1, f_2 on X such that

$$\mathrm{Im}\, f_i \subseteq (\mathbb{R} \times \{i\}) \sqcup \{\pm\infty\} \text{ and } f_1^{-1}(-\infty) = f_2^{-1}(-\infty).$$

We denote by $f: X \to B$ a bipath function.





*$f_1^{-1}(-\infty) = f_2^{-1}(-\infty)$ is needed to define a bipath sublevelset filtration.

Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath sublevelset filtration)

Let $f = (f_1, f_2)$ be a bipath function on a top. sp. X. For any b in B, let

$$(f \leq b) := \begin{cases} X & \text{if } b = +\infty \\ f_1^{-1}(\{-\infty\}) & \text{if } b = -\infty \\ f_1^{-1}([-\infty, r] \times \{1\}) & \text{if } b = (r, 1) \\ f_2^{-1}([-\infty, r] \times \{2\}) & \text{if } b = (r, 2) \end{cases}$$

Then, they give a functor $(f \leq \cdot\,)\colon B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

- $f\colon X \to B$: A bipath function.
- $V_f := H_q \circ (f \leq \cdot\,)\colon\ B \to \mathrm{vect}_{\mathrm{k}}$: Bipath PH of the sublevelset filtration of f.
- $\mathcal{B}(V_f)$: The bipath persistence diagram of f.

Idea of proof (Stability theorem)

- f, g: bipath functions with (◆).

$$f_1^{-1}(\{-\infty\}) \overset{\textbf{def}}{=} f_2^{-1}(\{-\infty\}) \overset{\blacklozenge}{=} g_1^{-1}(\{-\infty\}) \overset{\textbf{def}}{=} g_2^{-1}(\{-\infty\})$$

<u>**Sketch**</u>

$\epsilon := \ ||f, g||_B \quad \overset{(\blacklozenge)}{\rightsquigarrow}$

$$\begin{tikzcd}
(f \leq b) \arrow[r] \arrow[dr] & (f \leq \Lambda_\epsilon(b)) \arrow[r] \arrow[dr] & (f \leq \Lambda_{2\epsilon}(b)) \\
(g \leq b) \arrow[r] \arrow[ur] & (g \leq \Lambda_\epsilon(b)) \arrow[r] \arrow[ur] & (g \leq \Lambda_{2\epsilon}(b)),
\end{tikzcd}$$

Homology functor $\rightsquigarrow$

$$\begin{tikzcd}
V(f)_b \arrow[r] \arrow[dr] & V(f)_{\Lambda_\epsilon(b)} \arrow[r] \arrow[dr] & V(f)_{\Lambda_{2\epsilon}(b)} \\
V(g)_b \arrow[r] \arrow[ur] & V(g)_{\Lambda_\epsilon(b)} \arrow[r] \arrow[ur] & V(g)_{\Lambda_{2\epsilon}(b)}.
\end{tikzcd}$$

$\rightsquigarrow \quad V_f \sim_\epsilon V_g$

$\rightsquigarrow \quad d_{\mathrm{I}}(V_f, V_g) \leq \epsilon.$

$\rightsquigarrow \quad d_{\mathrm{B}}(\mathcal{B}(V_f), \mathcal{B}(V_g)) \overset{\text{Isometry thm.}}{=} d_{\mathrm{I}}(V_f, V_g) \leq \epsilon = ||f, g||_B.$ ■