Bipath Persistent homology and its stability

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Reference: S. Tada. Stability of Bipath Persistence Diagrams. arXiv: 2503.01614, 2025.

The aim is to introduce

Bipath persistent homology

which is an extension of persistent homology with a visualization (bipath persistence diagram) and

stability properties

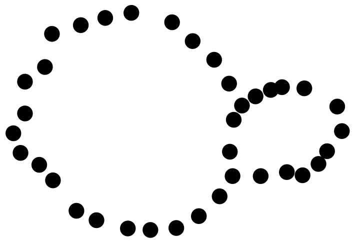
Contents

(1)Introduction: Persistent homology and related settings.

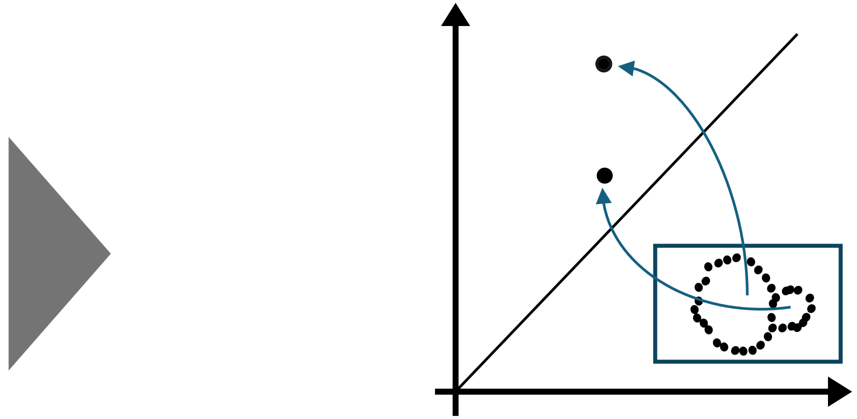
(2)Stability property of bipath persistence: Isometry theorem.

Introduction

- *Persistent homology* (PH) is a tool in Topological Data Analysis.
- It captures the persistence of “shape”(connected components, holes or voids) of data by a *persistence diagram* (PD).



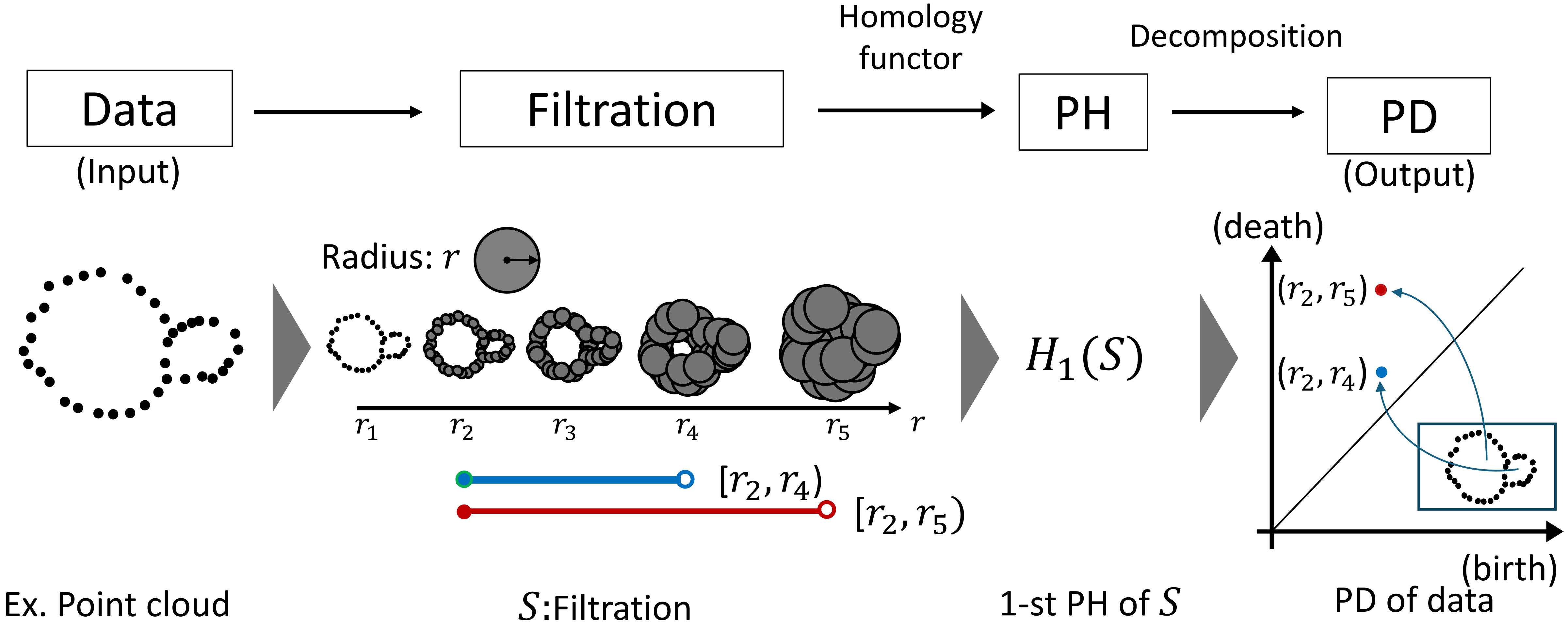
Ex. point cloud



Persistence diagram

PH: Persistent homology
PD: Persistent diagram

Introduction



Introduction

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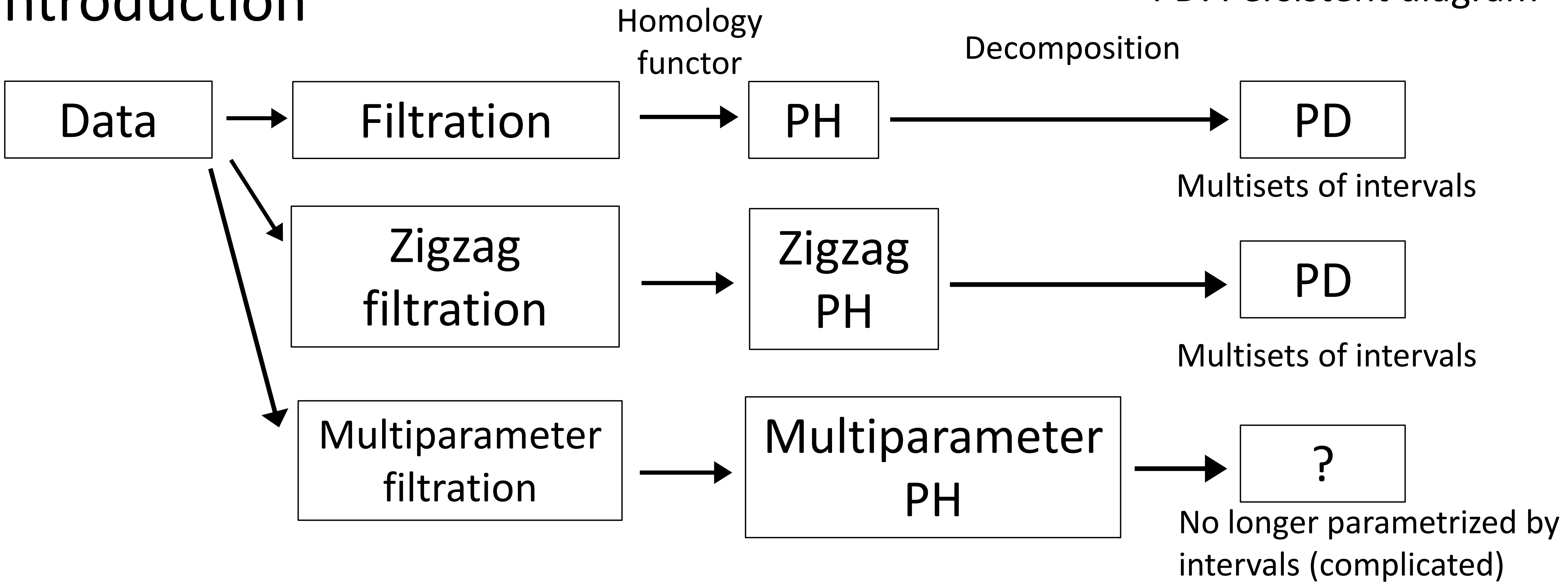
PD: Persistent diagram



Introduction

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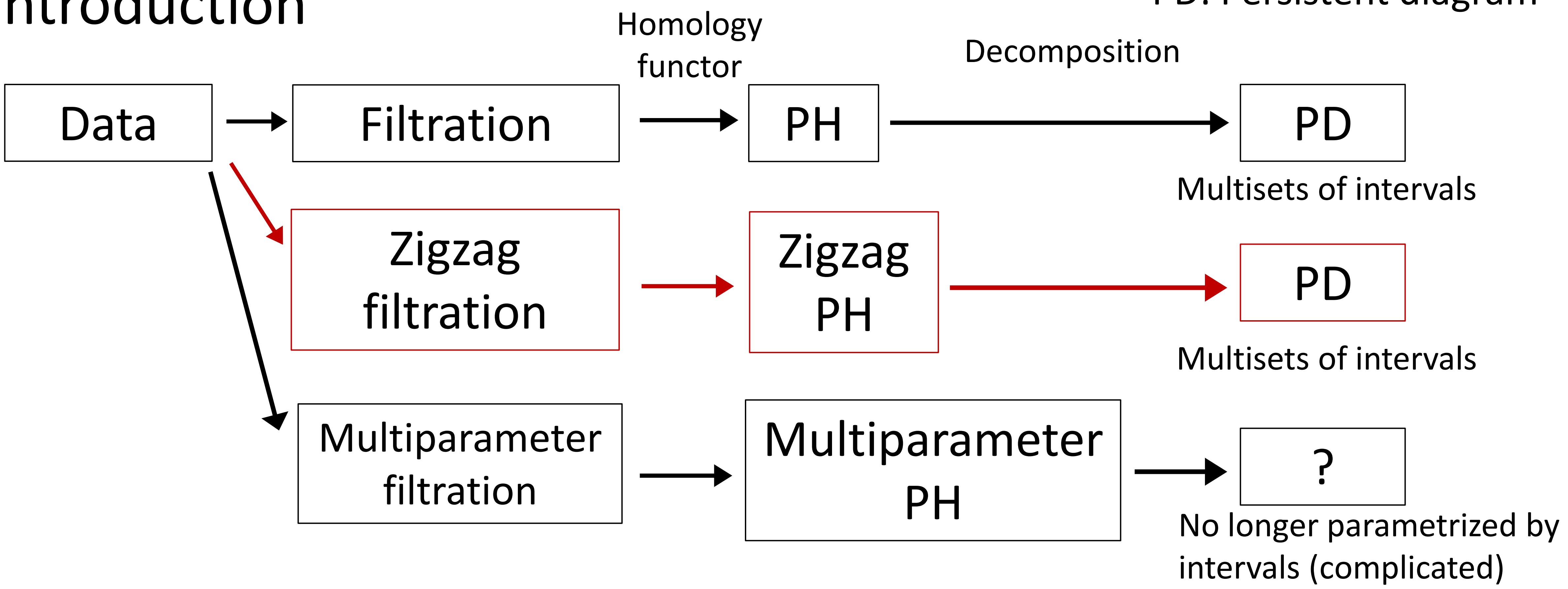


$$S: S_1 \supset S_2 \subset S_3 \supset S_4 \subset S_5$$

Zigzag filtration

$$H_i(S) = H_i(S_{r_1}) \leftarrow H_i(S_{r_2}) \rightarrow H_i(S_{r_3}) \leftarrow H_i(S_{r_4}) \rightarrow H_i(S_{r_5})$$

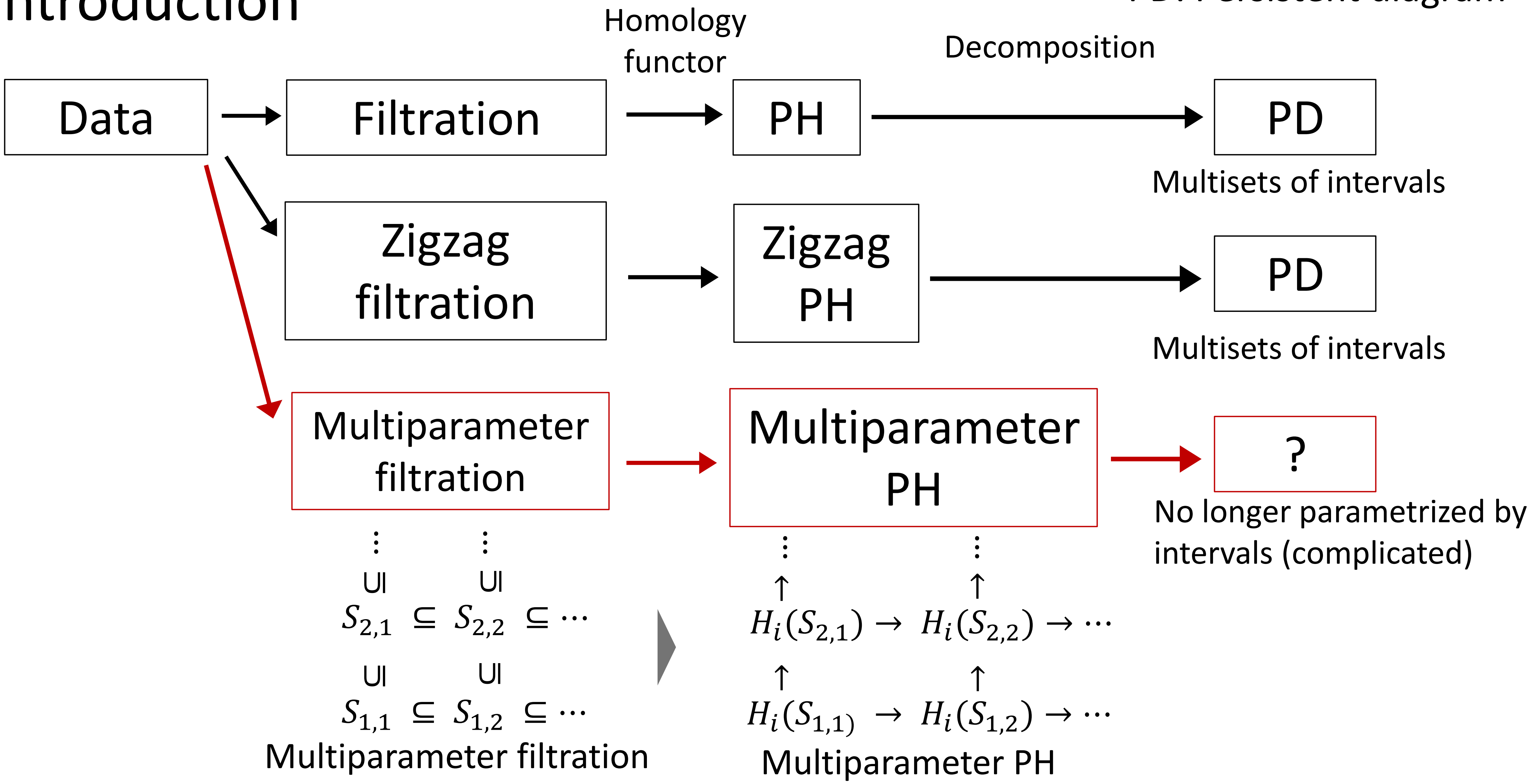
(Interval-decomposable)

Zigzag PH

Gunnar Carlsson, and Vin De Silva. Zigzag persistence. Foundations of computational mathematics 10 (2010): 367-405.

PH: Persistent homology
PD: Persistent diagram

Introduction



Gunnar Carlsson, and Afra Zomorodian. The Theory of Multidimensional Persistence. *Discrete Comput Geom* **42**, 71–93 (2009).

PH: Persistent homology
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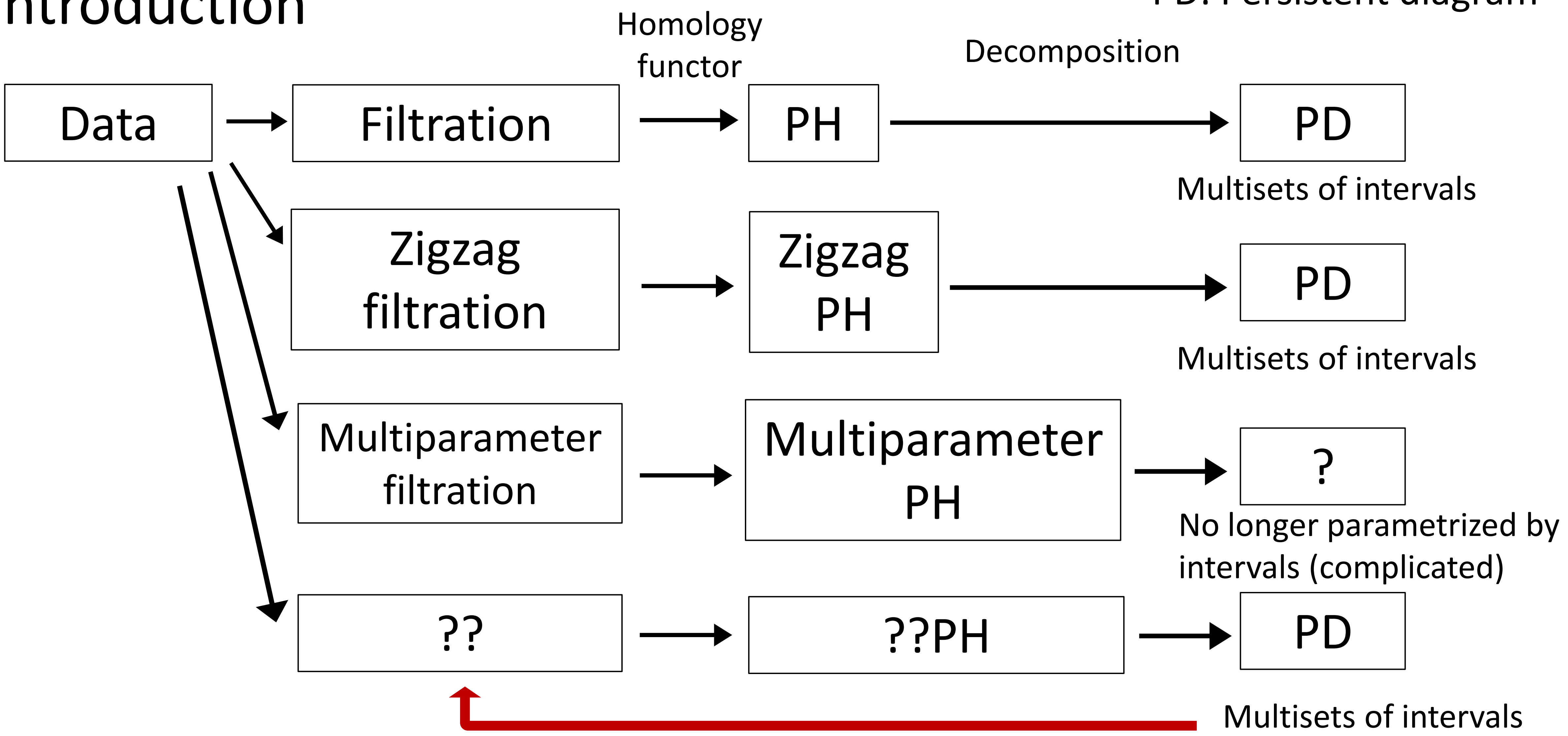
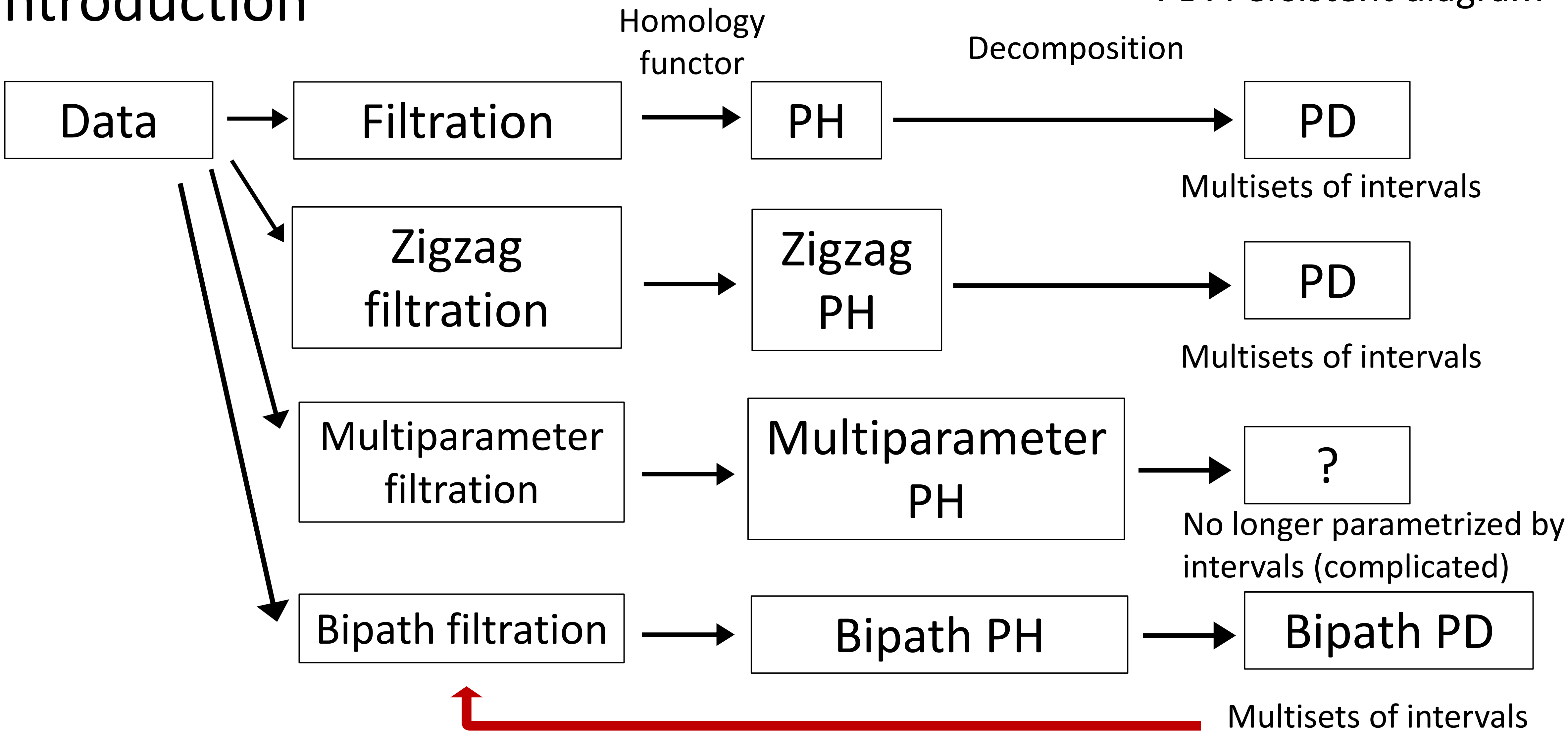


Do we have other arrangement of spaces like standard/zigzag filtration?

Introduction

PH: Persistent homology
PD: Persistent diagram



We propose bipath persistent homology as a new framework.

Introduction

Theorem [Aoki-Escolar-T, 25]

Let P be a connected finite poset. The following are equivalent.

(a) Every P-persistence module V is interval-decomposable.
(b) The Hasse diagram of P is one of the following forms:

$$A_n(a):\ 1 \longleftrightarrow \cdots \longleftrightarrow n$$

Zigzag posets (type A)

$$B_{n,m}:\ \begin{array}{ccccccc} & & & \leq & & & \\ & & 1 \rightarrow & 2 & \rightarrow \cdots \rightarrow n & & \\ & \nearrow & & & & \searrow & \\ -\infty & & & & & & +\infty \\ & \searrow & & \leq & & \nearrow & \\ & & 1' \rightarrow & 2' & \rightarrow \cdots \rightarrow m' & & \end{array}$$

Bipath posets

$x \leftrightarrow y$ represents either $x \rightarrow y$ or $x \leftarrow y$.

Introduction

We can consider a *bipath persistent homology* (bipath PH) of a *bipath filtration*, which can capture topological features across the filtration.

$$
\begin{array}{ccccccccc}
 & \nearrow & S_1 & \hookrightarrow & S_2 & \hookrightarrow \cdots \hookrightarrow & S_n & \searrow & \\
S_{-\infty} & & & & & & & & S_\infty \\
 & \searrow & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow \cdots \hookrightarrow & S_{m'} & \nearrow &
\end{array}
$$

Bipath filtration

$$
\begin{array}{ccccccccc}
 & \nearrow & H_q(S_1;k) & \longrightarrow & H_q(S_2;k) & \longrightarrow \cdots \longrightarrow & H_q(S_n;k) & \searrow & \\
H_q(S_{-\infty};k) & & & & & & & & H_q(S_\infty;k) \\
 & \searrow & H_q(S_{1'};k) & \longrightarrow & H_q(S_{2'};k) & \longrightarrow \cdots \longrightarrow & H_q(S_{m'};k) & \nearrow &
\end{array}
$$

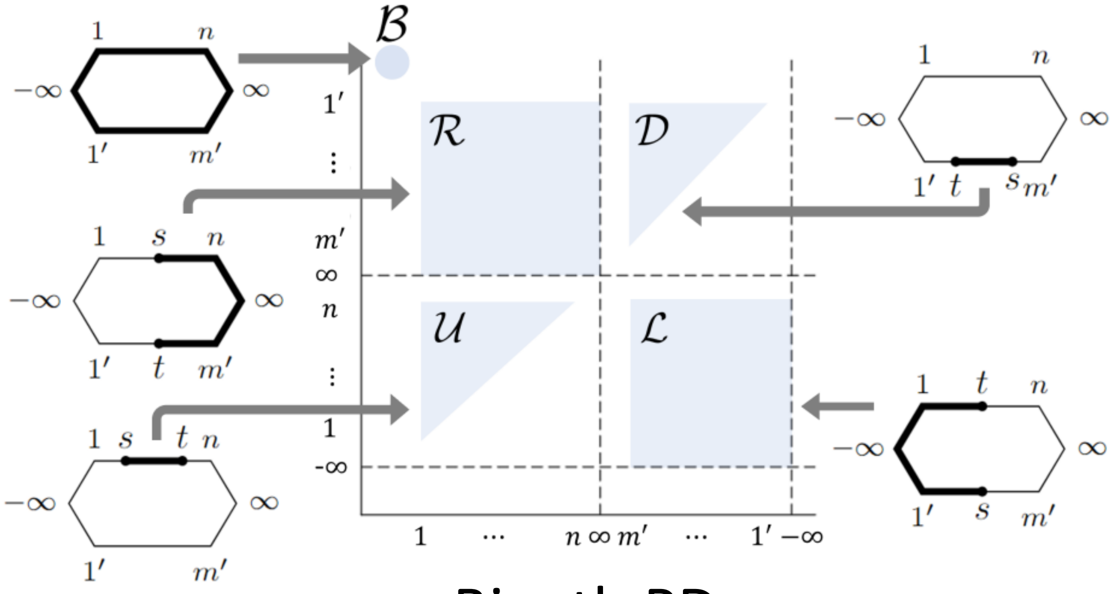
Bipath PH

Introduction

We can get a *Bipath Persistence Diagram* (Bipath PD).

$$\begin{array}{ccccccccc} & & H_q(S_1;k) & \longrightarrow & H_q(S_2;k) & \longrightarrow \cdots \longrightarrow & H_q(S_n;k) & & \\ & \nearrow & & & & & & \searrow & \\ H_q(S_{-\infty};k) & & & & & & & & H_q(S_{\infty};k) \\ & \searrow & & & & & & \nearrow & \\ & & H_q(S_{1'};k) & \longrightarrow & H_q(S_{2'};k) & \longrightarrow \cdots \longrightarrow & H_q(S_{m'};k) & & \end{array}$$

Interval decomposition



Bipath PD

T. Aoki, E. G. Escolar, and S. Tada. Bipath persistence. Japan Journal of Industrial and Applied Mathematics, 42:453–486, 2025.

Introduction

A recent study on bipath persistent homology:

	Bipath PH
Interval-Decomposability	○
Visualization(Bipath PD)	○
Algorithm(Implementation)	○
Stability theorem for Bipath PD	○
Application	-

Algorithm: Aoki, T., Escolar, E.G. & Tada, S. Bipath persistence. *Japan J. Indust. Appl. Math.* **42**, 453–486 (2025).
Implementation: https://github.com/ShunsukeTada1357/Bipathposets

Introduction

A recent study on bipath persistent homology:

	Bipath PH
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Algorithm(Implementation)	○
Stability theorem for Bipath PD	○
Application	-

Data → Bi-filtration → Bipath filtration → Bipath PD

$$
\begin{array}{ccccccc}
S_{1,4} & \rightarrow & S_{2,4} & \rightarrow & S_{3,4} & \rightarrow & S_{4,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,3} & \rightarrow & S_{2,3} & \rightarrow & S_{3,3} & \rightarrow & S_{4,3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,2} & \rightarrow & S_{2,2} & \rightarrow & S_{3,2} & \rightarrow & S_{4,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,1} & \rightarrow & S_{2,1} & \rightarrow & S_{3,1} & \rightarrow & S_{4,1}
\end{array}
$$

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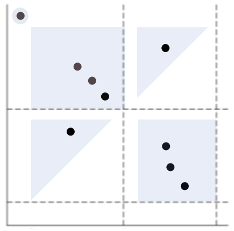

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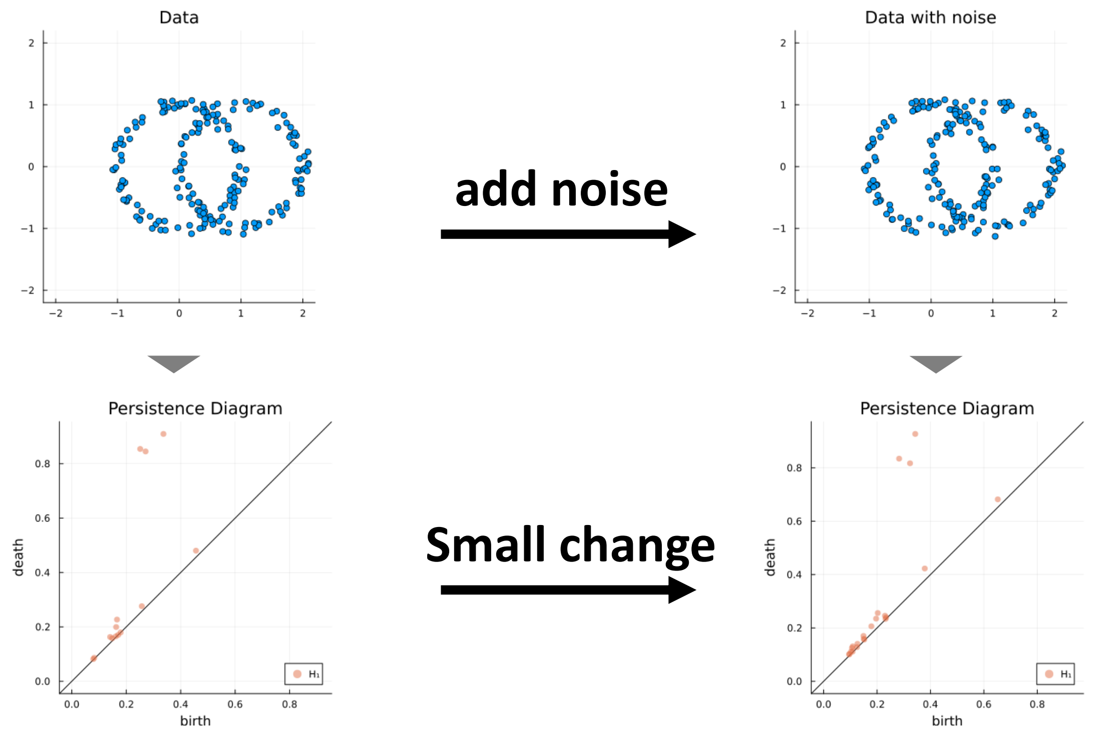
Stability of bipath persistence diagrams

Background: Stability theorem for standard PH

・ In persistent homology analysis stability theorem [Cohen-Steiner–Edelsbrunner–Harer '07] is important.

・ This implies small changes in data implies small changes in the PD.
⇝ It justifies the use of PH for studying noisy data.

Example



Cohen-Steiner, D., Edelsbrunner, H. and Harer, J. Stability of Persistence Diagrams. *Discrete Comput Geom* **37**, 103–120 (2007).

Background: Stability theorem for standard PH

Recall that stability theorem can be deduced by the isometry theorem.

Isometry theorem [Lesnick '15]

Let V and W be $\mathbb{R}$-persistence modules. Then, V and W are ϵ-interleaved if and only if there exist an ϵ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. Thus, we have

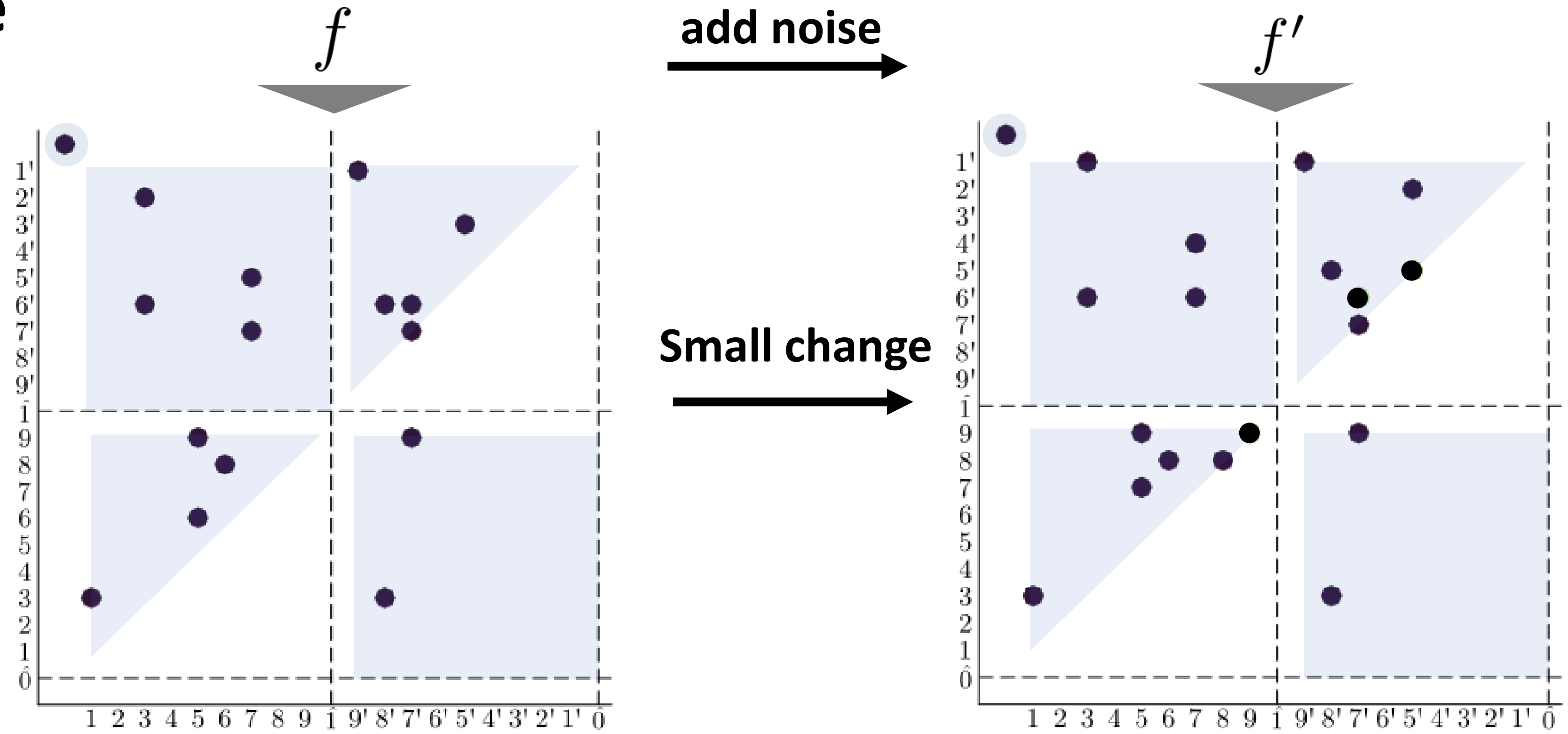
$$d_{\mathrm{B}}\big(\mathcal{B}(V), \mathcal{B}(W)\big) = d_{\mathrm{I}}(V, W).$$

Michael Lesnick. The theory of the interleaving distance on multidimensional persistence modules. Found. Comput. Math. 2015.

Stability theorem for bipath

Stability of bipath PDs holds [T ’25, Theorem 4.1].

Example



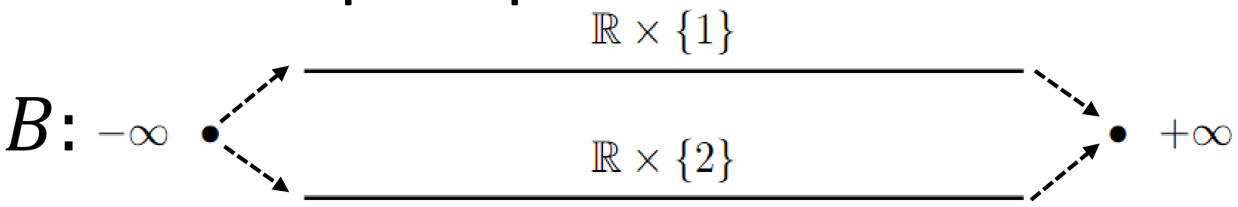
・This suggests a justification for using bipath PH on noisy data.

・This can be deduced by the <u>isometry theorem</u> for bipath persistence.

Stability theorem for bipath

To discuss stability, we consider continuous bipath poset B.

$$B_{n,m}\colon \begin{array}{ccccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ -\infty & & +\infty \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array} \quad \rightsquigarrow$$



$(\mathbb{R} \times \{1\}) \sqcup (\mathbb{R} \times \{2\}) \sqcup \{-\infty, +\infty\}$

Isometry theorem for bipath persistence [T ’25]

Let V and W be B-persistence modules. Then V and W are ϵ-interleaved if and only if there exist an ϵ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. Thus, we have $d_{\mathrm{B}}(\mathcal{B}(V), \mathcal{B}(W)) = d_{\mathrm{I}}(V, W)$.

$\rightsquigarrow$

- Setting for the definitions of d_{I} and d_{B}.
- Graph-theoretic approach in the general setting.
- Return to the bipath setting.

Stability theorem: Setting

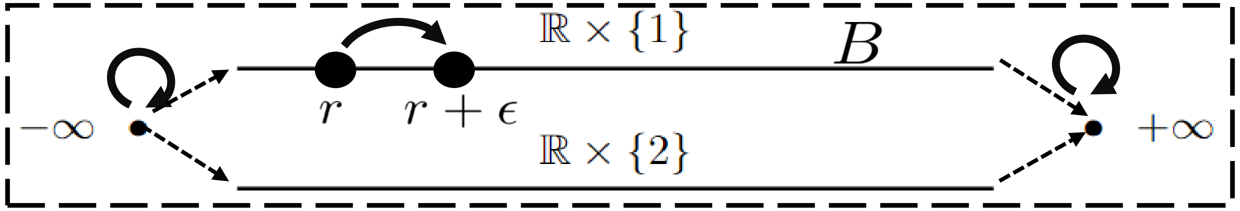
- Let k be a field, and let P be a poset.
- A P-persistence module is an object in $\mathrm{rep}_k(P) \coloneqq \mathrm{Fun}(P, \mathrm{vect}_k)$.
- For $V \cong \bigoplus_{\gamma\in\Gamma} V_\gamma \in \mathrm{rep}_k(P)$ (V_γ: indecomposable), set $\mathcal{B}(V)$:={ V_γ $|\gamma \in \Gamma$}.
- A translation on P is an order-preserving map h: $P \to P$ s. t. $p \leq h(p)$ for every $p \in P$.
- Fix a family of translations $\Lambda \coloneqq \{\Lambda_\epsilon\}_{\epsilon\in\mathbb{R}_{\geq 0}}$ on P satisfying:

$$\Lambda_0 = \mathrm{id}_P \text{ and } \Lambda_{\epsilon+\zeta} = \Lambda_\epsilon \circ \Lambda_\zeta \text{ for all } \epsilon,\ \zeta \in \mathbb{R}_{\geq 0}.$$

Example [T, ’25, Definition 3.4].

Let B be the bipath poset. We define Λ^B_ϵ :={Λ^B_ϵ}$_{\epsilon\in\mathbb{R}_{\geq 0}}$ by

$\Lambda^B_\epsilon(\pm\infty) \coloneqq \pm\infty$, and $\Lambda^B_\epsilon((r,i)) := (r+\epsilon, i)$ for $(r,i) \in \mathbb{R}\times\{i\}$ $(i = 1,2)$.



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Interleaving and bottleneck distances are defined w. r. t. $\Lambda \coloneqq \{\Lambda_\epsilon\}_{\epsilon\in\mathbb{R}_{\geq 0}}$.

Stability theorem: Setting

Let V, W be P-persistence modules, and $\epsilon \geq 0$.

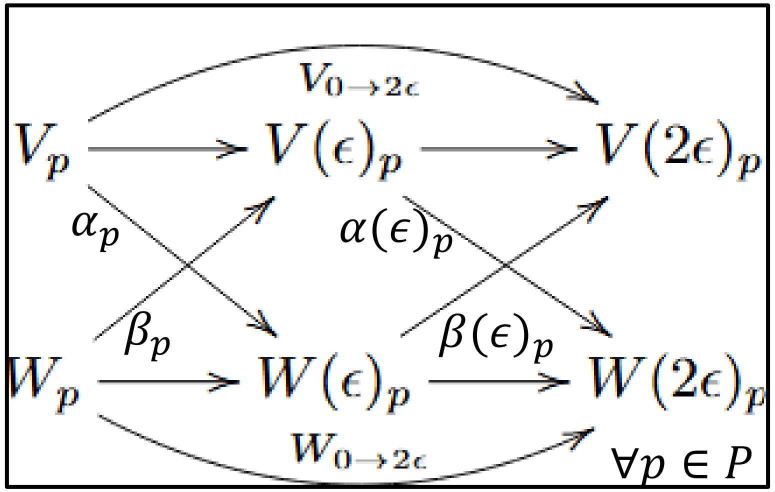
・We write $V(\epsilon) \coloneqq V \circ \Lambda_\epsilon \in \mathrm{rep}_k(P)$

(this gives a functor $(\cdot)(\epsilon)$: $\mathrm{rep}_k(P) \to \mathrm{rep}_k(P)$),

then, we have the induced morphism $V_{0\to\epsilon}\colon V \to V(\epsilon)$.

・We say that V and W are ϵ-interleaved and write $V \sim_\epsilon W$ if there is a pair of morphisms $\alpha\colon V \to W(\epsilon)$ and $\beta\colon W \to V(\epsilon)$ s. t.

$$V_{0\to 2\epsilon} = \beta(\epsilon) \circ \alpha \text{ and } W_{0\to 2\epsilon} = \alpha(\epsilon) \circ \beta.$$



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Definition (Interleaving distance)

The interleaving distance between P-persistence modules V and W is defined by $d_{\mathrm{I}}^{\Lambda}(V, W) := \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid V \sim_\epsilon W\}$.

Stability theorem: Setting

We say that a P-persistence module V is ϵ-trivial if $V_{0\to\epsilon} = 0$.

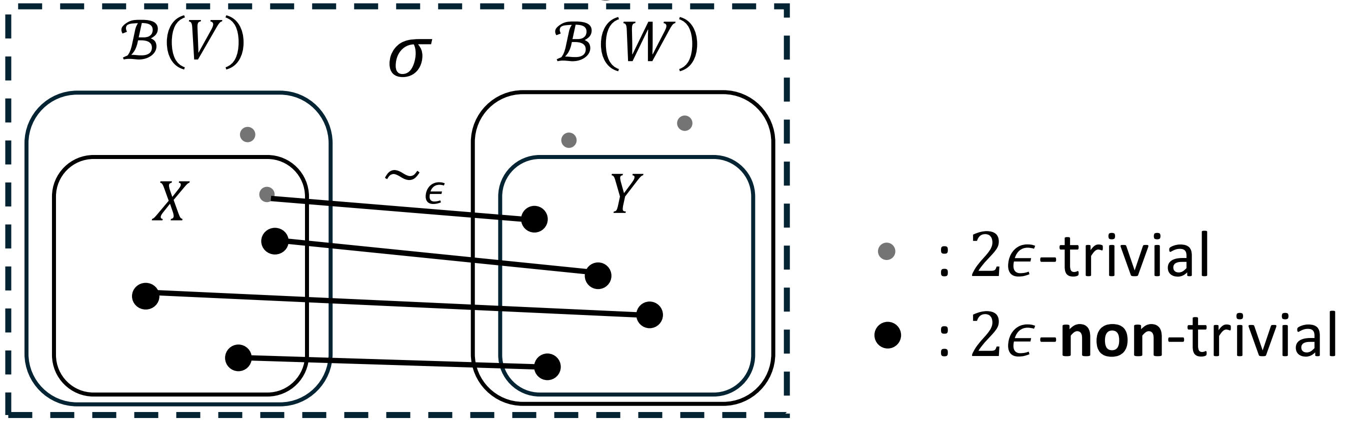
Definition (ϵ-matching)

Let V and W be P-persistence modules. An ϵ-matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$ is a partial matching $\sigma: \mathcal{B}(V) \supseteq X \overset{1:1}{\to} Y \subseteq \mathcal{B}(W)$ satisfying:

- Every $I \in (\mathcal{B}(V) \sqcup \mathcal{B}(W)) \setminus (X \sqcup Y)$ is 2ϵ-trivial.
- If $\sigma(I) = J$, then $I \sim_\epsilon J$.

We say that V and W are ϵ-matched if there is an ϵ-matching.

Picture of a ϵ-matching σ



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We say that V and W are ϵ-matched if there is an ϵ-matching.

Definition (Bottleneck distance)

The bottleneck distance between P-persistence modules V and W is defined by $d_{\mathrm{B}}^{\Lambda}(\mathcal{B}(V), \mathcal{B}(W)) := \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid V \text{ and } W \text{ are } \epsilon\text{-matched}\}$.

Stability theorem: Outline.

Remark

Let V and W be P-persistence modules. If V and W are ϵ-matched, then they are ϵ-interleaved. Thus, we have

$$d_{\mathrm{B}}^{\Lambda}(\mathcal{B}(V), \mathcal{B}(W)) \geq d_{\mathrm{I}}^{\Lambda}(V, W).$$

(∵ An ϵ-matching induces an ϵ-interleaving.)

$\rightsquigarrow$ We observe the converse:

・V and W are ϵ-interleaved. $\Rightarrow$ V and W are ϵ-matched.

Step1: Interpreting an ϵ-matching as a matching in a bipartite graph

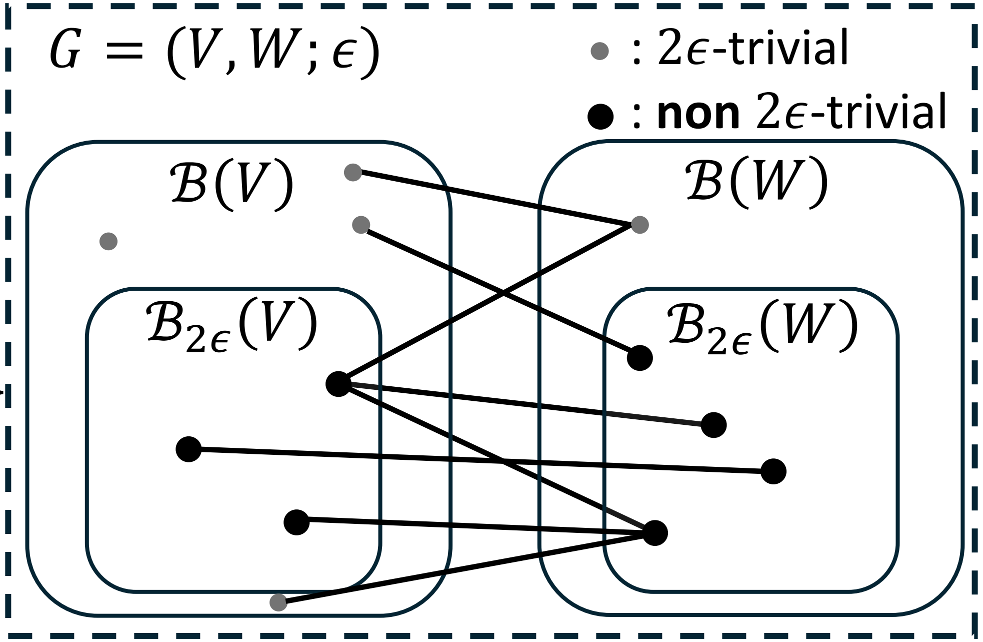
Step2: A sufficient condition for an ϵ-matching using a bipartite graph

Step3: Hall's marriage theorem is useful for showing the sufficient condition.

Step4: In the bipath setting, Step 2 is proved through Step 3.

Stability theorem: Outline Step 1

- Let V, W be P-persistence modules.
- Make bipartite graph $G = (V, W; \epsilon)$.
  - Vertices $\mathcal{B}(V) \sqcup \mathcal{B}(W)$
  - Edges $\{\{I, J\} \mid I \in \mathcal{B}(V), J \in \mathcal{B}(W), I \sim_\epsilon J\}$
- $\mathcal{B}_{2\epsilon}(V) \coloneqq \{I \in \mathcal{B}(V) \mid I$ is **not** 2ϵ-trivial$\}$



Proposition (1)[Bjerkevik '21, p.4]

The following are equivalent.

(a) V and W are ϵ-matched.

(b) $\exists$ a matching in $G = (V, W; \epsilon)$ that covers $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$.

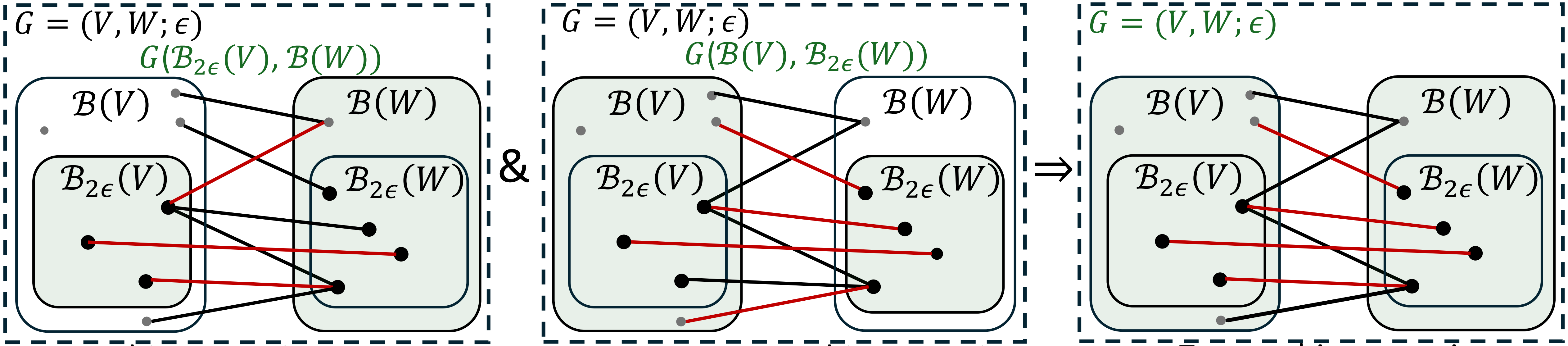
Håvard Bakke Bjerkevik. On the stability of interval decomposable persistence modules. Discrete &Computational Geometry, 66:92–121, 2021

Stability theorem: Outline Step 2

Proposition (2) [cf. Bjerkevik, '21, p. 111]

Let V and W be P-persistence modules. If the following are satisfied, then there is a matching in $G = (V, W; \epsilon)$ that covers $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$

- $\exists$ a matching in the full subgraph $G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$that covers $\mathcal{B}_{2\epsilon}(V)$.
- $\exists$ a matching in the full subgraph $G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ that covers $\mathcal{B}_{2\epsilon}(W)$.



$\exists$a matching covering $\mathcal{B}_{2\epsilon}(V)$

$\exists$a matching covering $\mathcal{B}_{2\epsilon}(W)$

$\exists$a matching covering $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$

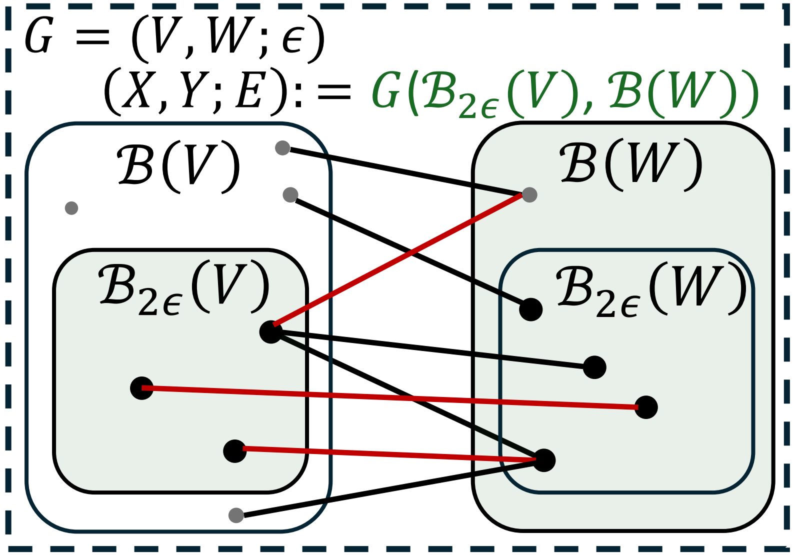
Stability theorem: Outline Step 3

Theorem (3) [Hall, 1935, Theorem 1]

Let $G = (X, Y; E)$ be a bipartite graph such that each vertex $x \in X$ has a finite neighborhood $N_G(x) \subseteq Y$. Then the following are equivalent:

(a) $\exists$ a matching in G that covers X.

(b) For every finite subset $X' \subseteq X$, we have $|X'| \leq |\cup_{x \in X'} N_G(x)|$.



<- Since $V \in \mathrm{rep}_k(P)$ is pointwise finite dimensional, $N_G(x) < \infty$ for every $x \in \mathcal{B}_{2\epsilon}(V)$[Bje21, p.110].

$\rightsquigarrow (X, Y; E) := G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$ satisfies the assumption of Hall's theorem.

$\rightsquigarrow$ Existence of a matching covering $\mathcal{B}_{2\epsilon}(V)$ is equivalent to (b):

$$\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(V), \text{ we have } |X'| \leq |\cup_{x \in X'} N_G(x)|.$$

Philip Hall. On representatives of subsets. Journal of the London Mathematical Society, s1-10(1):26–30,1935.

Stability theorem: Outline Step 3

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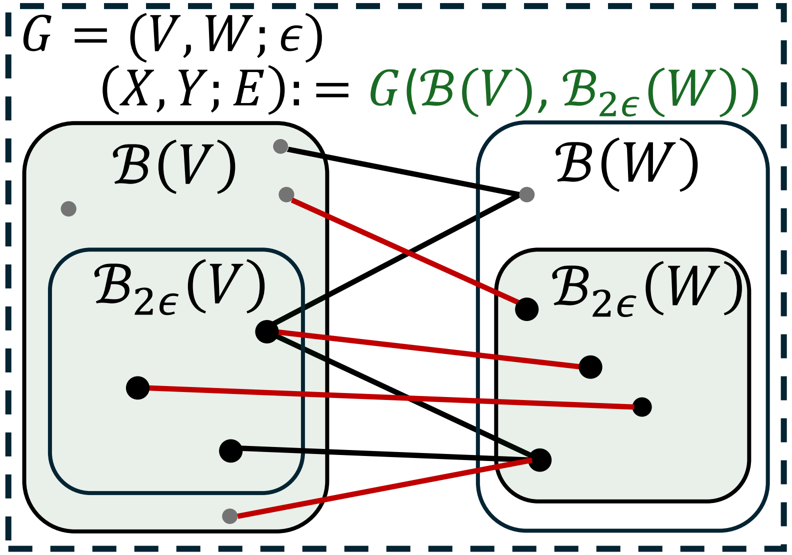

<- Since $W \in \mathrm{rep}_k(P)$ is pointwise finite dimensional, $N_G(x) < \infty$ for every $x \in \mathcal{B}_{2\epsilon}(W)$[Bje21, p.110].

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Stability theorem: Outline Step 1, 2, and 3

Let V and W be P-persistence modules.

V and W are ϵ-interleaved.

Cf. [Bje, '21, Ex. 5.3] $\not\Rightarrow$

- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(V)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.
- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(W)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.

Prop. (3) $\Leftrightarrow$ Hall

- $\exists$a matching in $G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$ that covers $\mathcal{B}_{2\epsilon}(V)$.
- $\exists$a matching in $G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ that covers $\mathcal{B}_{2\epsilon}(W)$.

Prop. (2) $\Rightarrow$

$\exists$a matching in $G = (V, W; \epsilon)$ that covers $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$.

Prop. (1) $\Leftrightarrow$

V and W are ϵ-matched.

Stability theorem: Outline Step 4

Let V and W be $\boldsymbol{B}$**-persistence modules**.

・B-persistence modules are interval-decomposable, with each interval determined by two elements of B.
・Λ_ϵ^B is a poset isomorphism ($\forall \epsilon \in \mathbb{R}_{\geq 0}$)

V and W are ϵ-interleaved

[T, '25] $\Rightarrow$

- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(V)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.
- $\forall X' \underset{\text{fin.}}{\subseteq} \mathcal{B}_{2\epsilon}(W)$, $|X'| \leq |\cup_{x \in X'} N_G(x)|$ holds.

Prop. (3) $\Leftrightarrow$ Hall

- $\exists$a matching in $G(\mathcal{B}_{2\epsilon}(V), \mathcal{B}(W))$ that covers $\mathcal{B}_{2\epsilon}(V)$.
- $\exists$a matching in $G(\mathcal{B}(V), \mathcal{B}_{2\epsilon}(W))$ that covers $\mathcal{B}_{2\epsilon}(W)$.

Prop. (2) $\Rightarrow$

$\exists$a matching in $G = (V, W; \epsilon)$ that covers $\mathcal{B}_{2\epsilon}(V) \sqcup \mathcal{B}_{2\epsilon}(W)$.

Prop. (1) $\Leftrightarrow$

V and W are ϵ-matched. $\rightsquigarrow d_B^{\Lambda_\epsilon^B}(\mathcal{B}(V), \mathcal{B}(W)) = d_I^{\Lambda_\epsilon^B}(V, W)$

Summary

・ We introduce bipath PH, which is an extension of standard PH.

・ Bipath PDs have stability (arXiv: 2503.01614), and it is shown using an isometry theorem.

Discussion

・ Application of bipath PH to real data. $\rightsquigarrow$ We recently discussed the use of it for image data analysis with material scientists.

・ Relation with interleaving distance for finite bipath posets by Alonso and Liu (arXiv: 2501.00322).

$\rightsquigarrow$ Universality of interleaving distance.

Thank you for your listening.