MathBio Workshop: Shape and Movement in Life Sciences

A Computation of Bipath Persistent Homology and Bipath Persistence Diagrams

Shunsuke Tada
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Self-Introduction

Name: Shunsuke Tada

Affiliation: Kobe University D3 (Supervisor: Emerson-sensei)

Research: **Persistent homology using representation theory**

Next Affiliation: Postdoc at Tohoku University (MathCCS)

Hobby: walking, running (Full-marathon experience), and others

An aim is talking about

Bipath persistent homology

which is an extension of persistent homology.

- A Visualization/Computation of Bipath Persistence Diagrams
  (Joint work with Toshitaka Aoki, Emerson G. Escolar)
- Stability of bipath persistence diagrams (bipath PDs)

・T. Aoki, E. G. Escolar, and S. Tada. Bipath persistence. Japan Journal of Industrial and Applied Mathematics, 42:453–486, 2025.
・S. Tada. Stability of Bipath Persistence Diagrams. arXiv: 2503.01614, 2025.

Contents

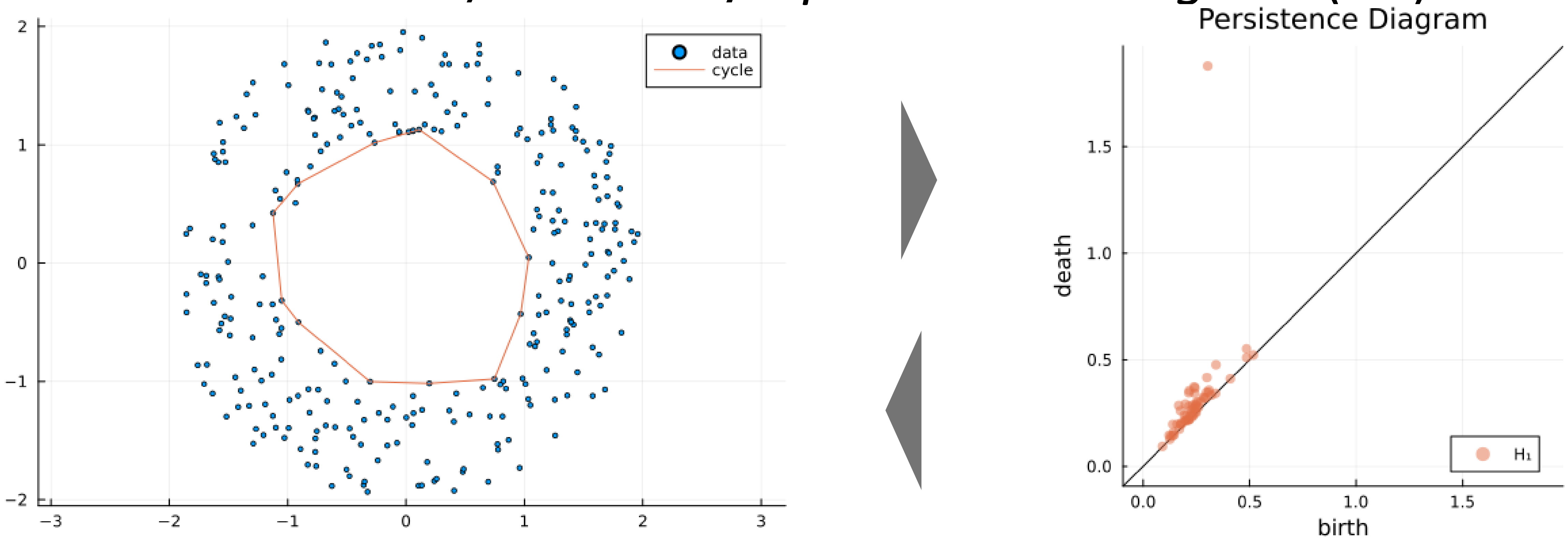
(1)Introduction: Persistent Homology and others

(2)Bipath persistence diagrams/Computation

(3)Stability of bipath persistence diagrams

Persistent homology

- *Persistent homology* (PH) is a tool in Topological Data Analysis.
- It captures the persistence of “shape”(connected components, holes or voids) of data by a *persistence diagram* (PD).



Applications

・ Material science

・ Evolutional biology

・ Cosmology (cosmic web)

・ Computational gastronomy

and others…

・Yasuaki Hiraoka Hiraoka, Takenobu Nakamura, Akihiko Hirata, Emerson G. Escolar, Kaname Matsue, and Yasumasa Nishiura. Hierarchical structures of amorphous solids characterized by persistent homology. *Proceedings of the National Academy of Sciences*, *113*(26), 7035-7040, 2016.

・Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. Topology of viral evolution. Proceedings of the National Academy of Sciences, 110(46):18566–18571, 2013

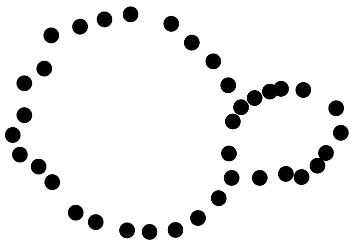
・Thierry Sousbie. The persistent cosmic web and its filamentary structure–I. theory and implementation. Monthly Notices of the Royal Astronomical Society, 414(1):350–383, 2011.
and others.

・Emerson G. Escolar, Yuta Shimada, and Masahiro Yuasa. A topological analysis of the space of recipes. International Journal of Gastronomy and Food Science, 39:101088, 2025.

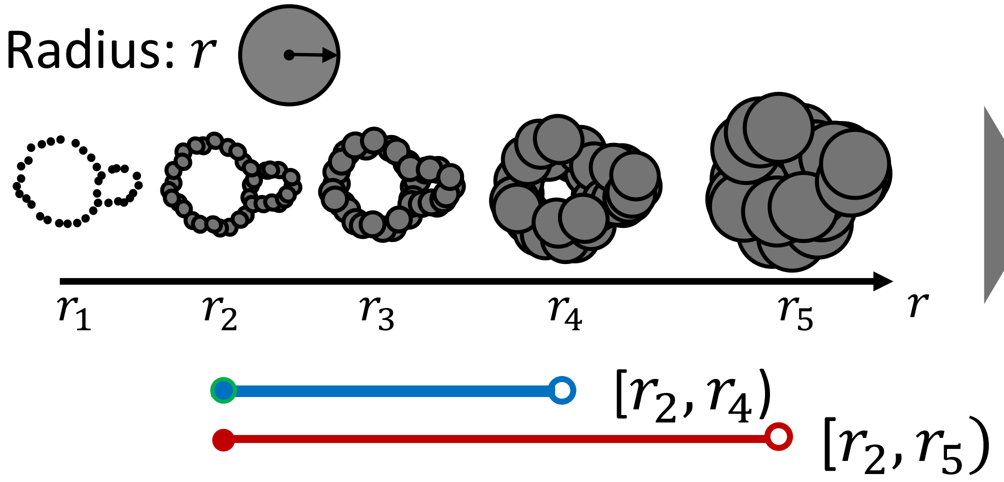
How to capture shape

PH: Persistent homology
PD: Persistent diagram

Data (Input) → Filtration —Homology functor→ PH —Decomposition→ PD (Output)



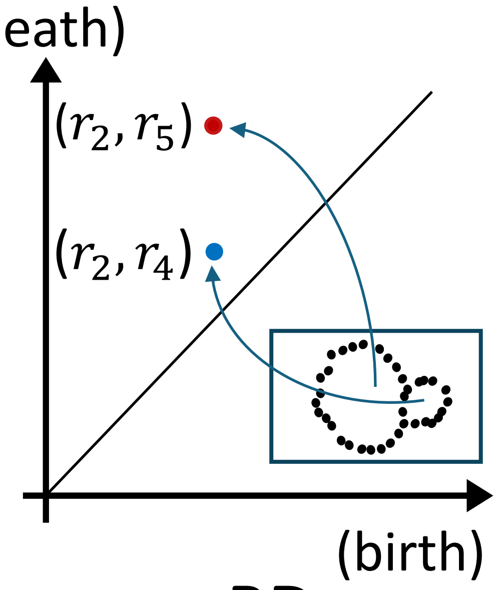
Ex. Point cloud



S:Filtration

$H_1(S)$

PH of S



PD

- Each point in PD corresponds to a hole.
- The upper left point in a PD corresponds to a larger hole.

PH formalizes the “persistence” of the holes.

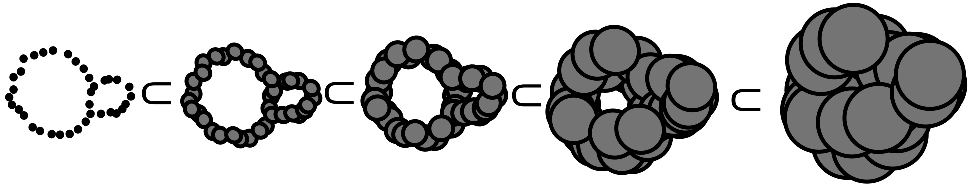
PH: Persistent homology
PD: Persistent diagram

How to capture shape

Filtration —Homology functor→ PH

- k-vector spaces
- Liner maps

S: $S_{r_1} \hookrightarrow S_{r_2} \hookrightarrow S_{r_3} \hookrightarrow S_{r_4} \hookrightarrow S_{r_5}$



$r_1 = 0$ r_2 r_3 r_4 r_5

$[r_2, r_4)$

$[r_2, r_5)$

S: Filtration

$$H_1(S_{r_1}) \to H_1(S_{r_2}) \to H_1(S_{r_3}) \to H_1(S_{r_4}) \to H_1(S_{r_5})$$

$$\cong$$

$$0 \to k^2 \to k^2 \to k \to 0$$

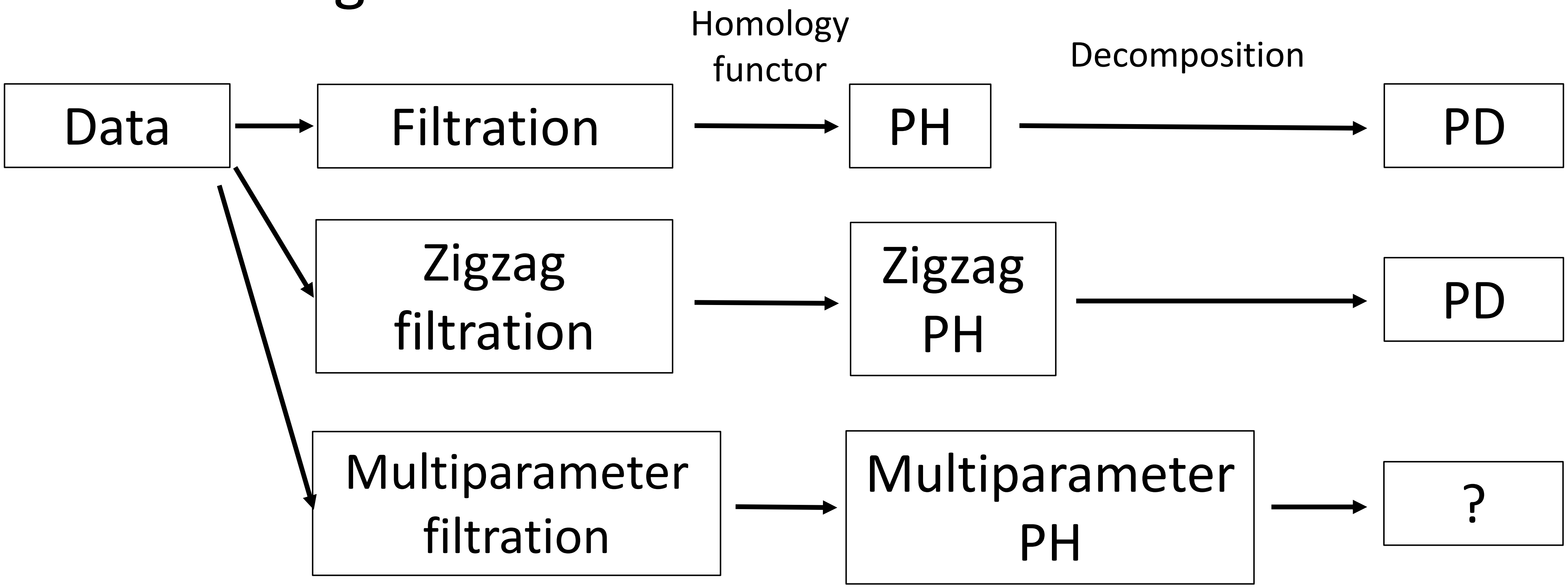
$\cong$ uniquely decomposed

$$I[r_2, r_3]: 0 \to k \xrightarrow{\text{id}} k \to 0 \to 0$$

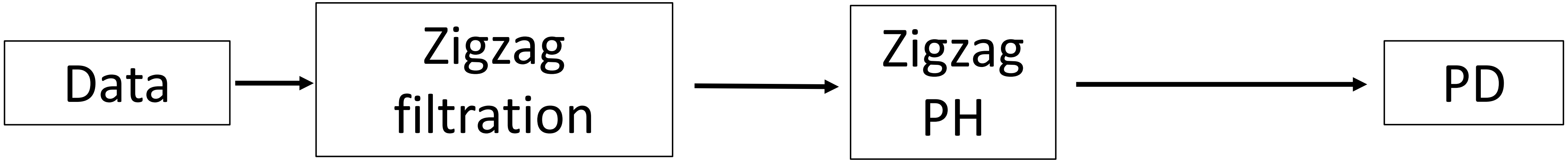
$$\oplus$$

$$I[r_2, r_4]: 0 \to k \xrightarrow{\text{id}} k \xrightarrow{\text{id}} k \to 0$$

Other settings



Other settings: zigzag PH

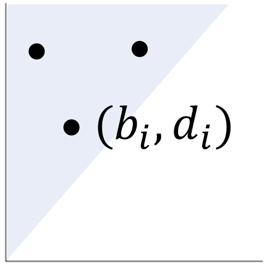


$$S\colon S_1 \supset S_2 \subset S_3 \supset S_4 \subset S_5$$

Zigzag filtration

$$H_i(S) = H_i(S_{r_1}) \leftarrow H_i(S_{r_2}) \rightarrow H_i(S_{r_3}) \leftarrow H_i(S_{r_4}) \rightarrow H_i(S_{r_5})$$
$$\cong \bigoplus I[b_i, d_i]^{m_{b_i,d_i}}$$ (Interval-decomposable)

e.g., Temporal network



PD

Gunnar Carlsson, and Vin De Silva. Zigzag persistence. Foundations of computational mathematics 10 (2010): 367-405.
Myers, A., Muñoz, D., Khasawneh, F. A., & Munch, E. (2023). Temporal network analysis using zigzag persistence. *EPJ Data Science*, *12*(1), 6.

Other settings: multiparameter PH

Data → Multiparameter filtration → Multiparameter PH → ?

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \cup\!\!\mid & & \cup\!\!\mid & & \\ S_{2,1} & \subseteq & S_{2,2} & \subseteq & \cdots \\ \cup\!\!\mid & & \cup\!\!\mid & & \\ S_{1,1} & \subseteq & S_{1,2} & \subseteq & \cdots \end{array}$$

S

Multiparameter filtration

- density
- time
- partial charge, and others…

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \\ H_i(S_{2,1}) & \to & H_i(S_{2,2}) & \to & \cdots \\ \uparrow & & \uparrow & & \\ H_i(S_{1,1)} & \to & H_i(S_{1,2}) & \to & \cdots \end{array}$$

$H_i(S)$

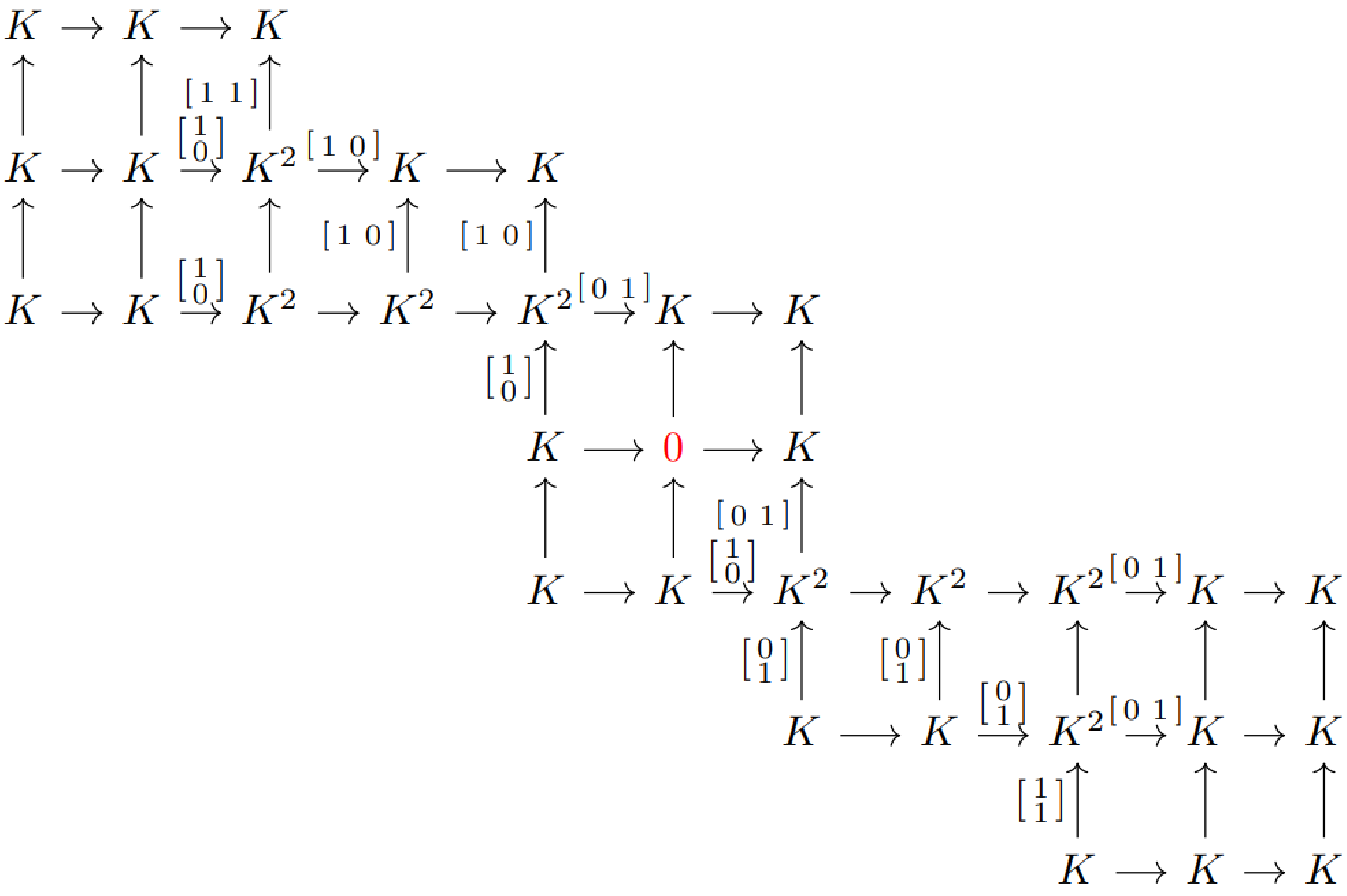
Multiparameter PH

$$H_i(S) \cong \bigoplus_{\lambda \in \Lambda} V_\lambda$$

No longer parametrized by intervals (complicated)

Gunnar Carlsson, and Afra Zomorodian. The Theory of Multidimensional Persistence. *Discrete Comput Geom* **42**, 71–93 (2009).

Difficulty of multiparameter PH



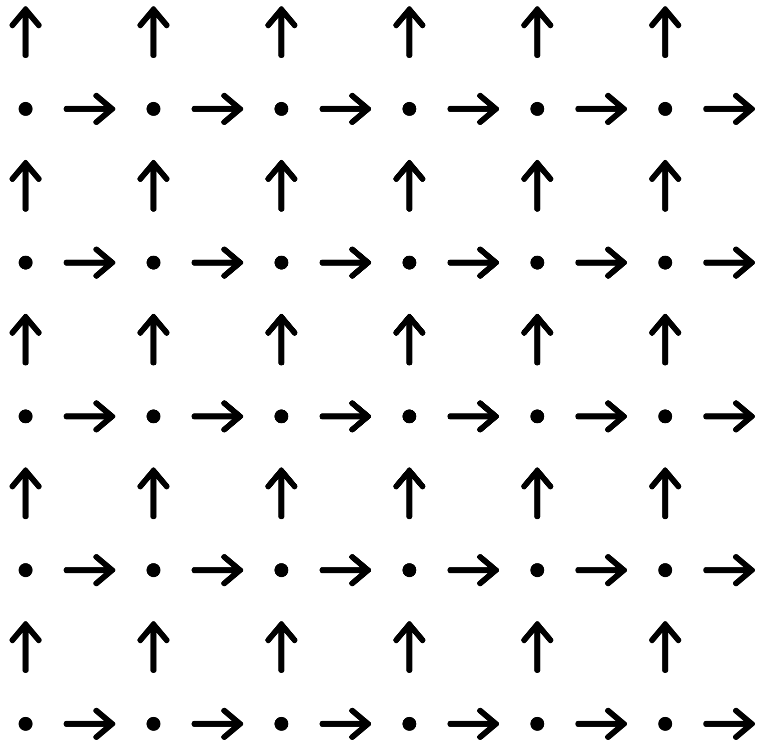
An indecomposable module:

This is not interval.

Mickaël Buchet, and Emerson G. Escolar. Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module. *Journal of Applied and Computational Topology.*

Difficulty of multiparameter PH

It is complicated to classify all the indecomposable module. (wild representation type)



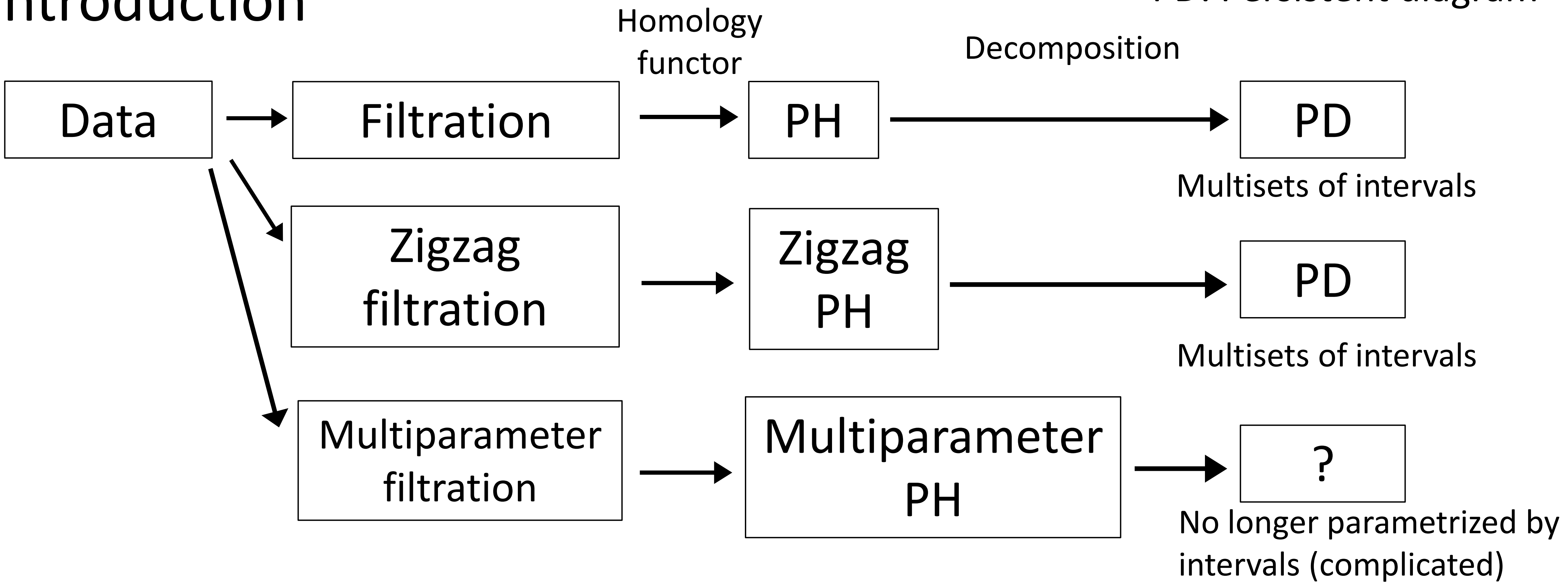
・Zbigniew Leszczyński. On the representation type of tensor product algebras. Fundamenta Mathematicae, 144(2):143–161, 1994.

・Zbigniew Leszczyński and Andrzej Skowroński. Tame triangular matrix algebras. Colloquium Mathematicum, 86(2):259 303, 2000.

・Ulrich Bauer, Magnus B. Botnan, Steffen Oppermann, and Johan Steen. Cotorsion torsion triples and the representation theory of filtered hierarchical clustering. Advances in Mathematics, 369:107171, 2020.

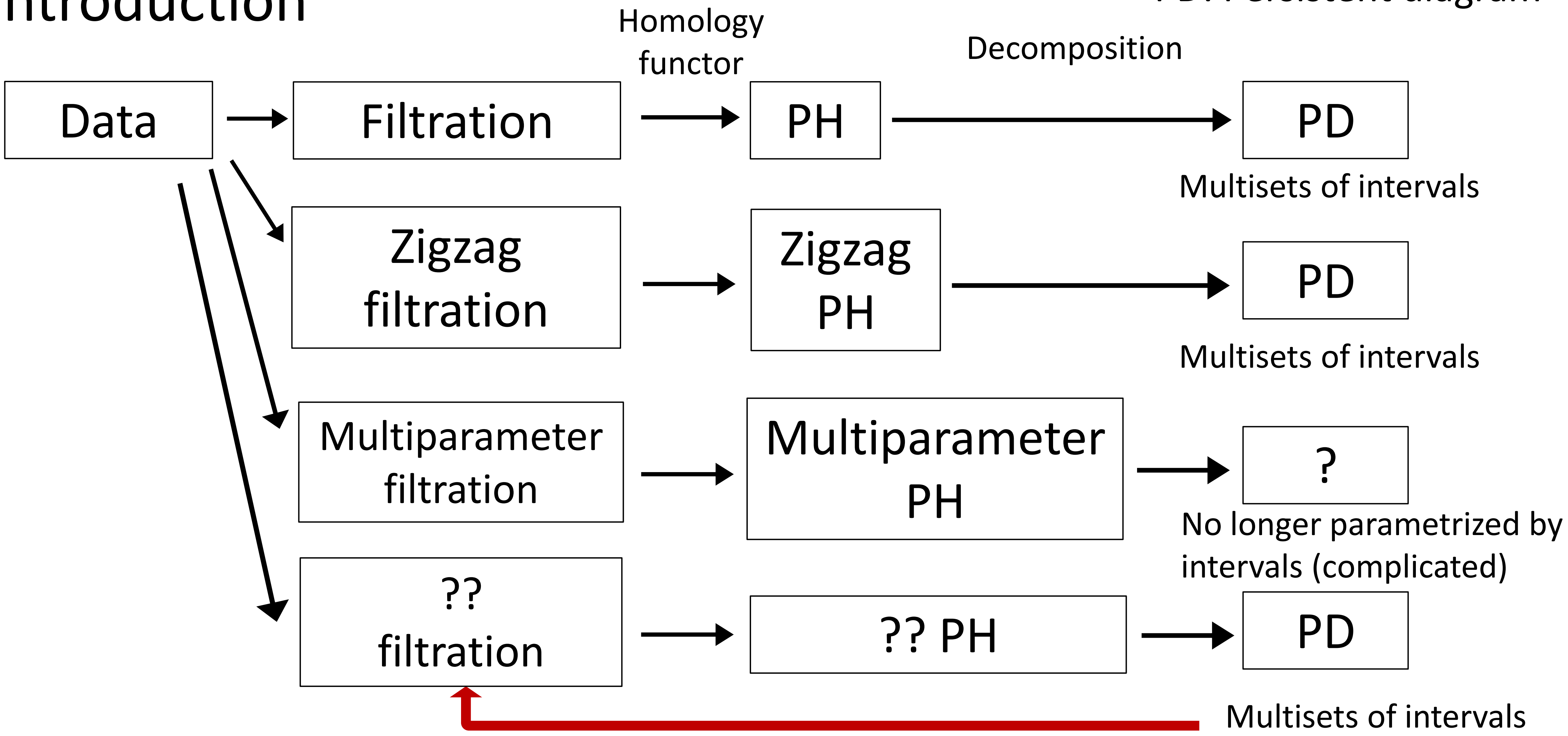
Introduction

PH: Persistent homology
PD: Persistent diagram



PH: Persistent homology
PD: Persistent diagram

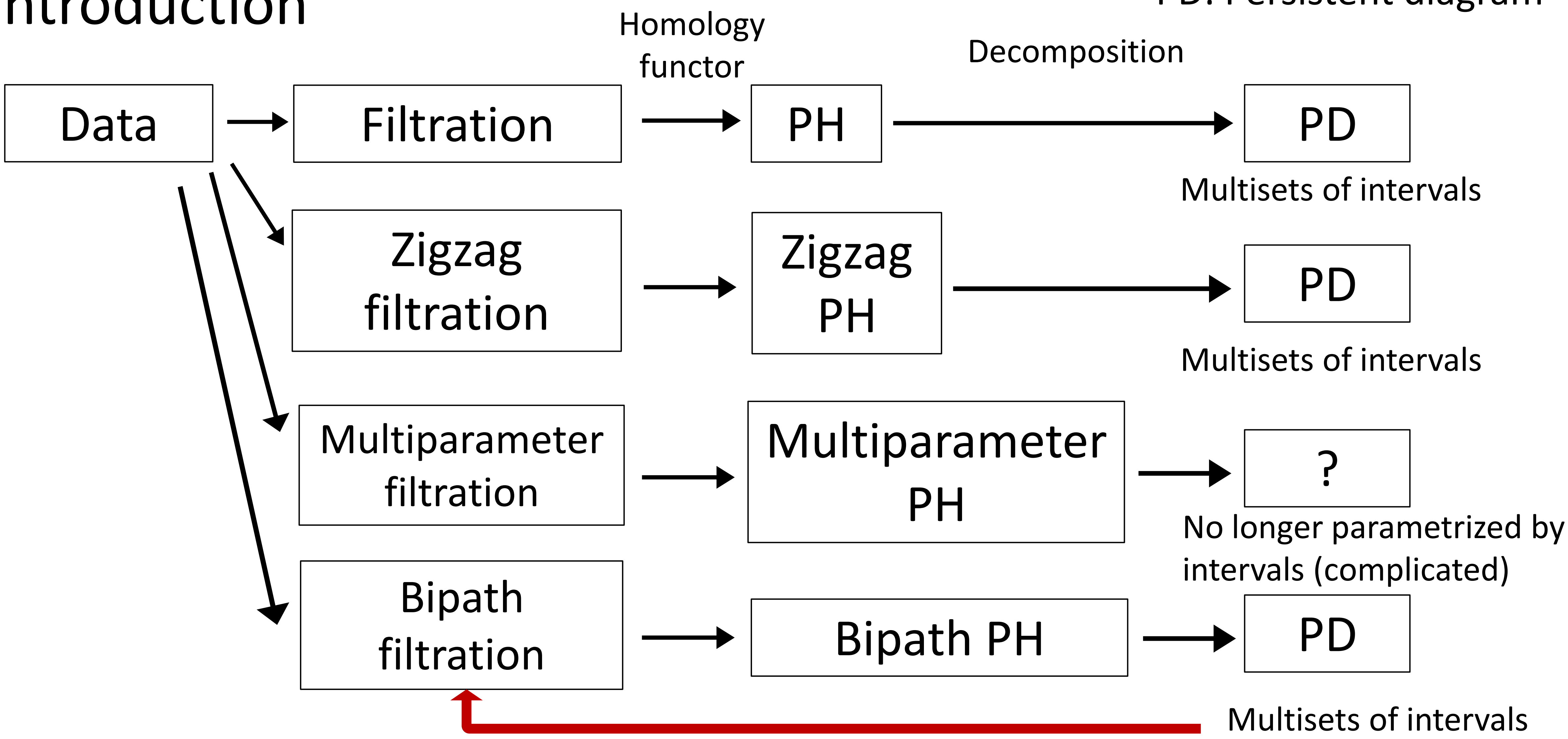
Introduction



Do we have other arrangement of spaces like standard/zigzag filtration?

Introduction

PH: Persistent homology
PD: Persistent diagram



Bipath filtration ([Aoki-Escolar-T, 25])

Introduction

$$S: \quad \begin{array}{ccccccccc} & & S_1 & \hookrightarrow & S_2 & \hookrightarrow \cdots \hookrightarrow & S_n & & \\ & \nearrow & & & & & & \searrow & \\ S_{\hat{0}} & & & & & & & & S_{\hat{1}} \\ & \searrow & & & & & & \nearrow & \\ & & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow \cdots \hookrightarrow & S_{m'} & & \end{array}$$

Bipath filtration

||

$$S: \quad \begin{array}{c} S_{\hat{0}} \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{\hat{1}} \\ \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ S_{\hat{0}} \hookrightarrow S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} \hookrightarrow S_{\hat{1}} \end{array}$$

A pair of two filtrations sharing the same spaces at the ends.

Introduction

We can consider a *bipath persistent homology* (bipath PH) of a filtration.

$$
\begin{array}{ccccccccc}
 & & S_1 & \hookrightarrow & S_2 & \hookrightarrow \cdots \hookrightarrow & S_n & & \\
 & \nearrow & & & & & & \searrow & \\
S_{\hat{0}} & & & & & & & & S_{\hat{1}} \\
 & \searrow & & & & & & \nearrow & \\
 & & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow \cdots \hookrightarrow & S_{m'} & &
\end{array}
$$

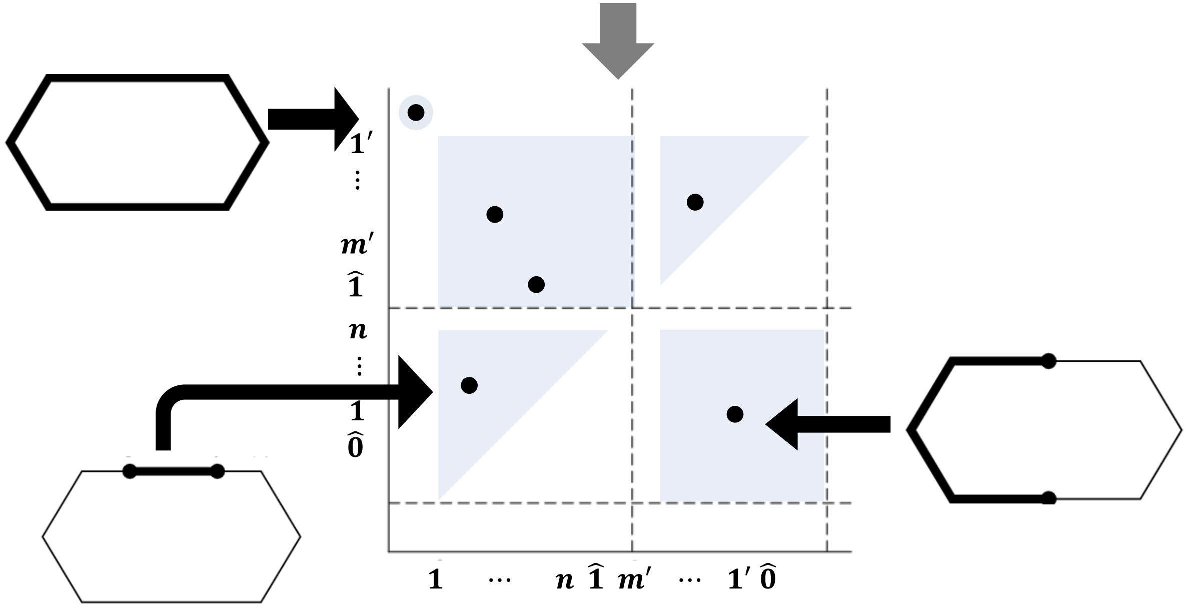
$$
\begin{array}{ccccccccc}
 & & H_q(S_1;k) & \longrightarrow & H_q(S_2;k) & \longrightarrow \cdots \longrightarrow & H_q(S_n;k) & & \\
 & \nearrow & & & & & & \searrow & \\
H_q(S_{\hat{0}};k) & & & & & & & & H_q(S_{\hat{1}};k) \\
 & \searrow & & & & & & \nearrow & \\
 & & H_q(S_{1'};k) & \longrightarrow & H_q(S_{2'};k) & \longrightarrow \cdots \longrightarrow & H_q(S_{m'};k) & &
\end{array}
$$

Bipath PH

Introduction

We can get a *Bipath Persistence Diagram* (Bipath PD).

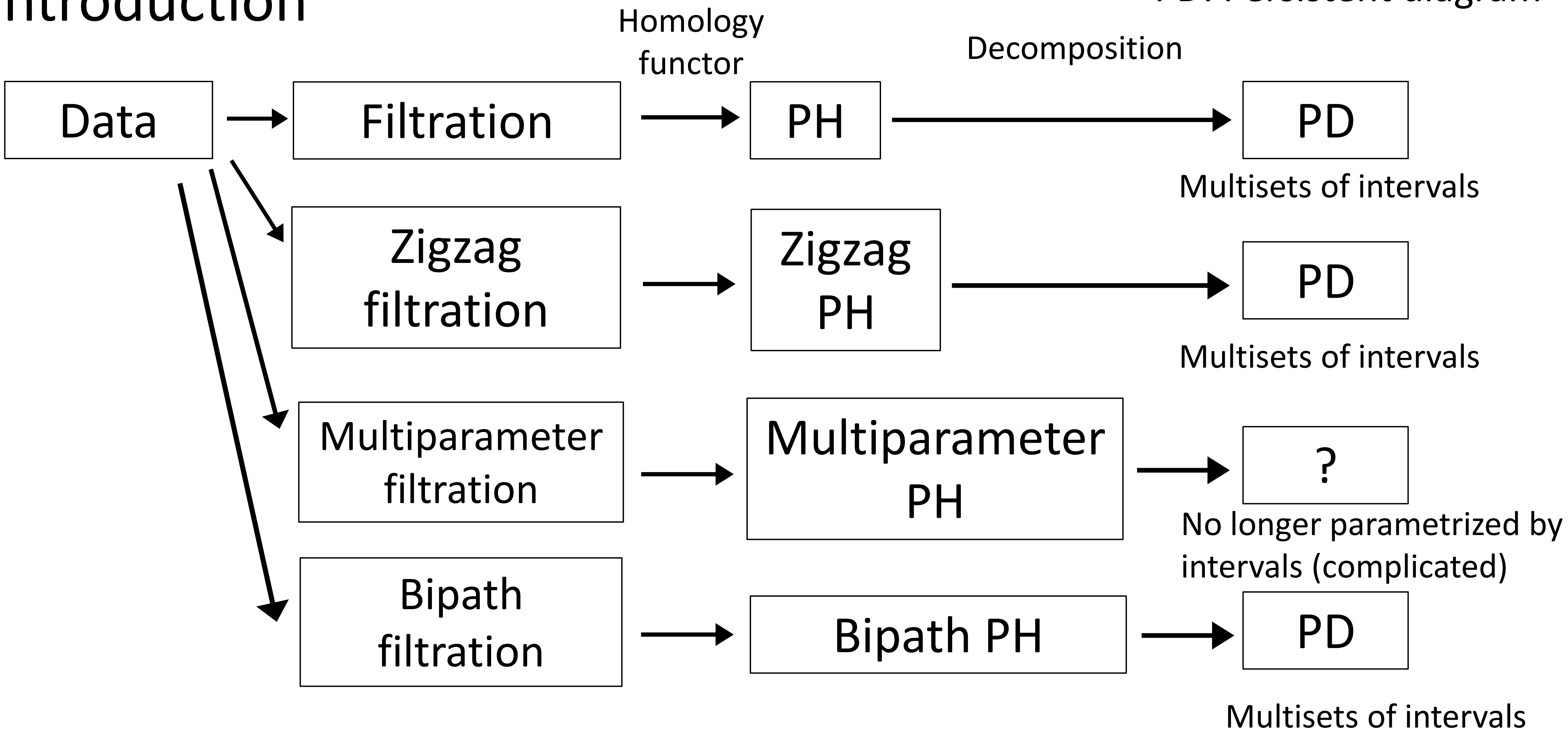
$$\begin{array}{ccccccccc} & & H_q(S_1;k) & \longrightarrow & H_q(S_2;k) & \longrightarrow \cdots \longrightarrow & H_q(S_n;k) & & \\ & \nearrow & & & & & & \searrow & \\ H_q(S_{\hat{0}};k) & & & & & & & & H_q(S_{\hat{1}};k) \\ & \searrow & & & & & & \nearrow & \\ & & H_q(S_{1'};k) & \longrightarrow & H_q(S_{2'};k) & \longrightarrow \cdots \longrightarrow & H_q(S_{m'};k) & & \end{array}$$



Bipath PD

Introduction

PH: Persistent homology
PD: Persistent diagram



->What can bipath PH do?

Introduction

Bipath PH can be used to…
(1) study the persistence of topological features across a pair of filtrations connected at their ends, to compare the two filtrations.

$$
S: \begin{array}{c} S_{\hat{0}} \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{\hat{1}} \\ \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ S_{\hat{0}} \hookrightarrow S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} \hookrightarrow S_{\hat{1}} \end{array}
$$

Introduction

Bipath PH can be used to…
(2) obtain an invariant for multiparameter PH.

$$
\begin{array}{ccccccc}
S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
\uparrow & & & & & & \uparrow \\
S_{1,3} & & & & & & S_{4,3} \\
\uparrow & & & & & & \uparrow \\
S_{1,2} & & & & & & S_{4,2} \\
\uparrow & & & & & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}
\quad \subseteq \quad
\begin{array}{ccccccc}
S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,3} & \to & S_{2,3} & \to & S_{3,3} & \to & S_{4,3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,2} & \to & S_{2,2} & \to & S_{3,2} & \to & S_{4,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}
$$

Bipath filtration. Bifiltration.

Introduction

Bipath PH can be used to…
(2) obtain an invariant for multiparameter PH.

$$
\begin{array}{ccccccc}
S_{1,4} & \rightarrow & S_{2,4} & \rightarrow & S_{3,4} & \rightarrow & S_{4,4} \\
\uparrow & & & & & & \uparrow \\
S_{1,3} & & & & S_{3,3} & \rightarrow & S_{4,3} \\
\uparrow & & & & \uparrow & & \\
S_{1,2} & & S_{2,2} & \rightarrow & S_{3,2} & & \\
\uparrow & & \uparrow & & & & \\
S_{1,1} & \rightarrow & S_{2,1} & & & &
\end{array}
\quad \subseteq \quad
\begin{array}{ccccccc}
S_{1,4} & \rightarrow & S_{2,4} & \rightarrow & S_{3,4} & \rightarrow & S_{4,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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Bipath filtration. Bifiltration.

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 & & S_{2,4} & \rightarrow & S_{3,4} & \rightarrow & S_{4,4} \\
 & & \uparrow & & & & \uparrow \\
S_{1,3} & \rightarrow & S_{2,3} & & S_{3,3} & \rightarrow & S_{4,3} \\
\uparrow & & & & \uparrow & & \\
S_{1,2} & & S_{2,2} & \rightarrow & S_{3,2} & & \\
\uparrow & & \uparrow & & & & \\
S_{1,1} & \rightarrow & S_{2,1} & & & &
\end{array}
\quad \subseteq \quad
\begin{array}{ccccccc}
S_{1,4} & \rightarrow & S_{2,4} & \rightarrow & S_{3,4} & \rightarrow & S_{4,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,3} & \rightarrow & S_{2,3} & \rightarrow & S_{3,3} & \rightarrow & S_{4,3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,2} & \rightarrow & S_{2,2} & \rightarrow & S_{3,2} & \rightarrow & S_{4,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,1} & \rightarrow & S_{2,1} & \rightarrow & S_{3,1} & \rightarrow & S_{4,1}
\end{array}
$$

Bipath filtration. Bifiltration.

Introduction

Bipath PH can be used to…
(2) obtain an invariant for multiparameter PH.

$$
\begin{array}{ccccccc}
 & & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
 & & \uparrow & & & & \uparrow \\
S_{1,3} & \to & S_{2,3} & & & & S_{4,3} \\
\uparrow & & & & & & \uparrow \\
S_{1,2} & & & & & & S_{4,2} \\
\uparrow & & & & & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}
\quad \subseteq \quad
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\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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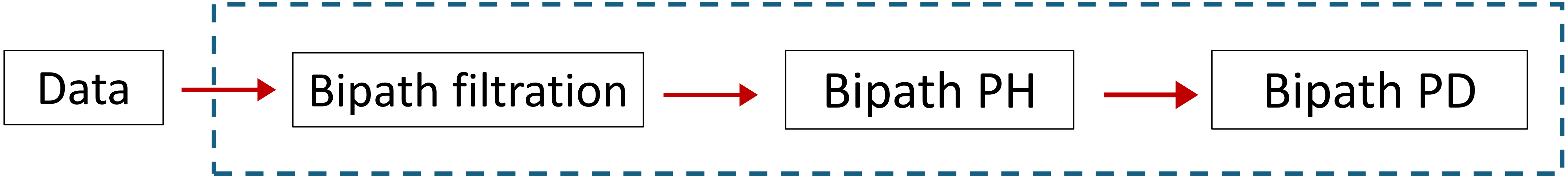
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,2} & \to & S_{2,2} & \to & S_{3,2} & \to & S_{4,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}
$$

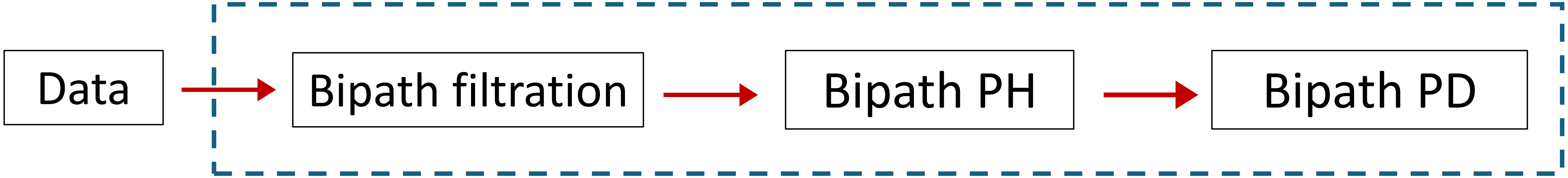
Bipath filtration. Bifiltration.

Introduction



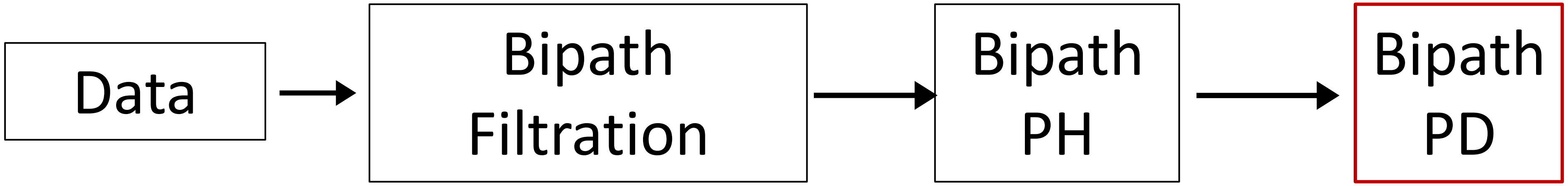
		Bipath PH
	Interval-Decomposability	O
(1)	Visualization(Bipath PD)	O
(2)	Algorithm(Implementation)	O
(3)	Stability theorem for Bipath PD	O
	Application	-

Introduction



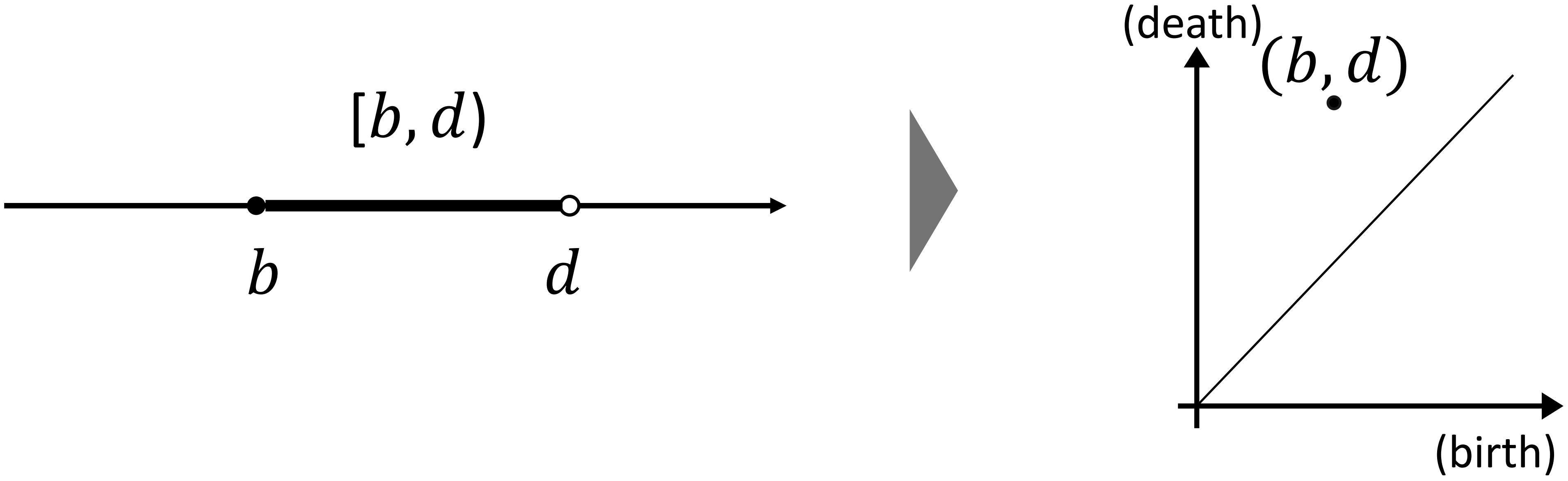
		Bipath PH
	Interval-Decomposability	○
(1)	Visualization(Bipath PD)	○
(2)	Algorithm(Implementation)	○
(3)	Stability theorem for Bipath PD	○
	Application	-

Bipath Persistence Diagram
(Bipath PD)



How to visualize.

Recall: standard persistence diagram



In the standard setting, intervals are visualized through the correspondence between intervals and points on the plane.

$$[b, d) \mapsto (b, d) \in \mathbb{R}^2$$

Bipath Persistence

Definition Bipath poset

Let m and n be non-negative numbers. A *bipath poset* $B_{n,m}$ is a poset consisting of two totally ordered sets

$$1 \leq 2 \leq \cdots \leq n, \text{ and } 1' \leq 2' \leq \cdots \leq m'$$

with the global minimum and the global maximum

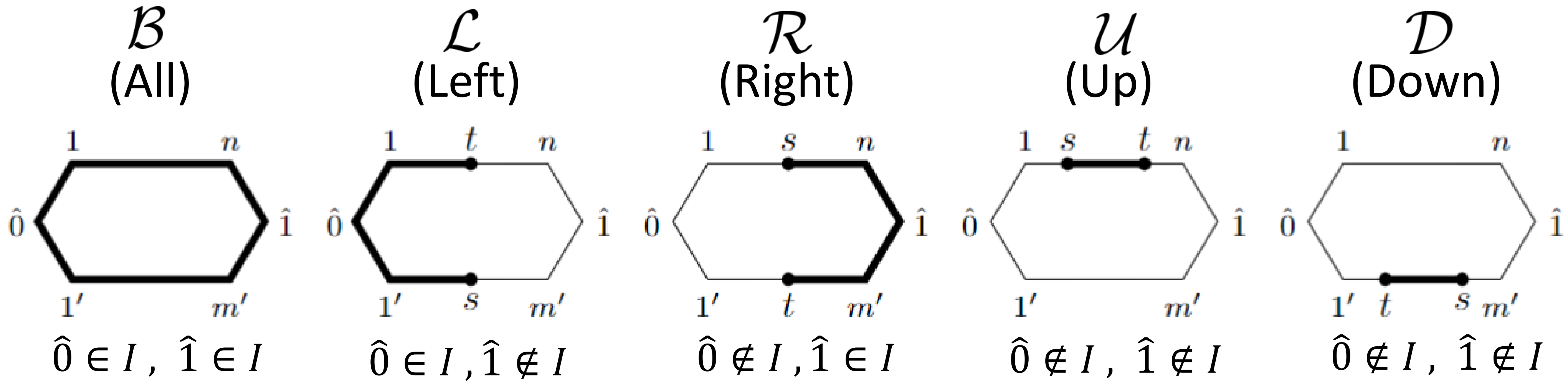
$$\hat{0} \text{ and } \hat{1}.$$

The Hasse diagram:

$$B_{n,m}: \quad \begin{array}{ccc} & 1 \rightarrow 2 \xrightarrow{\leq} \cdots \rightarrow n & \\ \nearrow & & \searrow \\ \hat{0} & & \hat{1} \\ \searrow & & \nearrow \\ & 1' \rightarrow 2' \xrightarrow{\leq} \cdots \rightarrow m' & \end{array}$$

Bipath Persistence

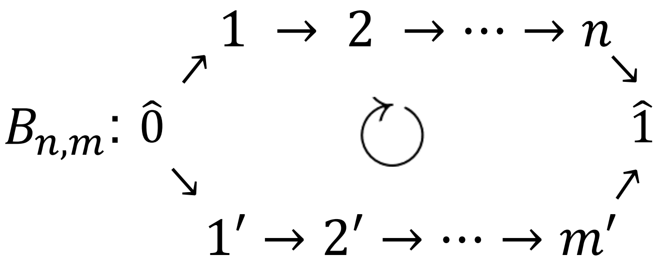
Intervals in $B := B_{n,m}$ are one of the following forms:



・Each interval in B (except for B) is written by the pair $\langle s, t \rangle$ $(s, t \in B)$ by taking end points of the interval.

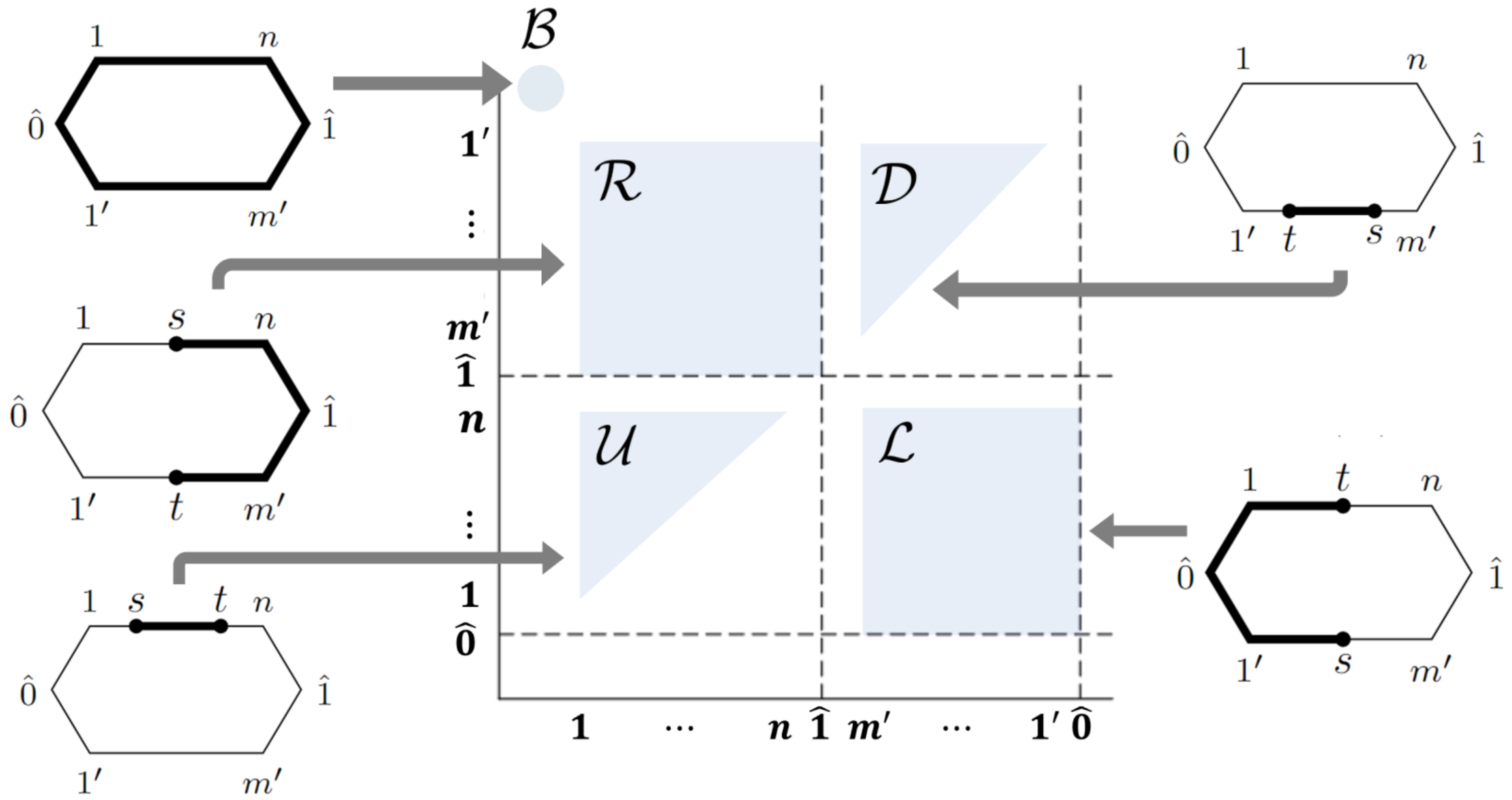
A correspondence of intervals and points

(1)Put elements of $B_{n,m}$ (in clockwise) on the vertical and horizontal axes.



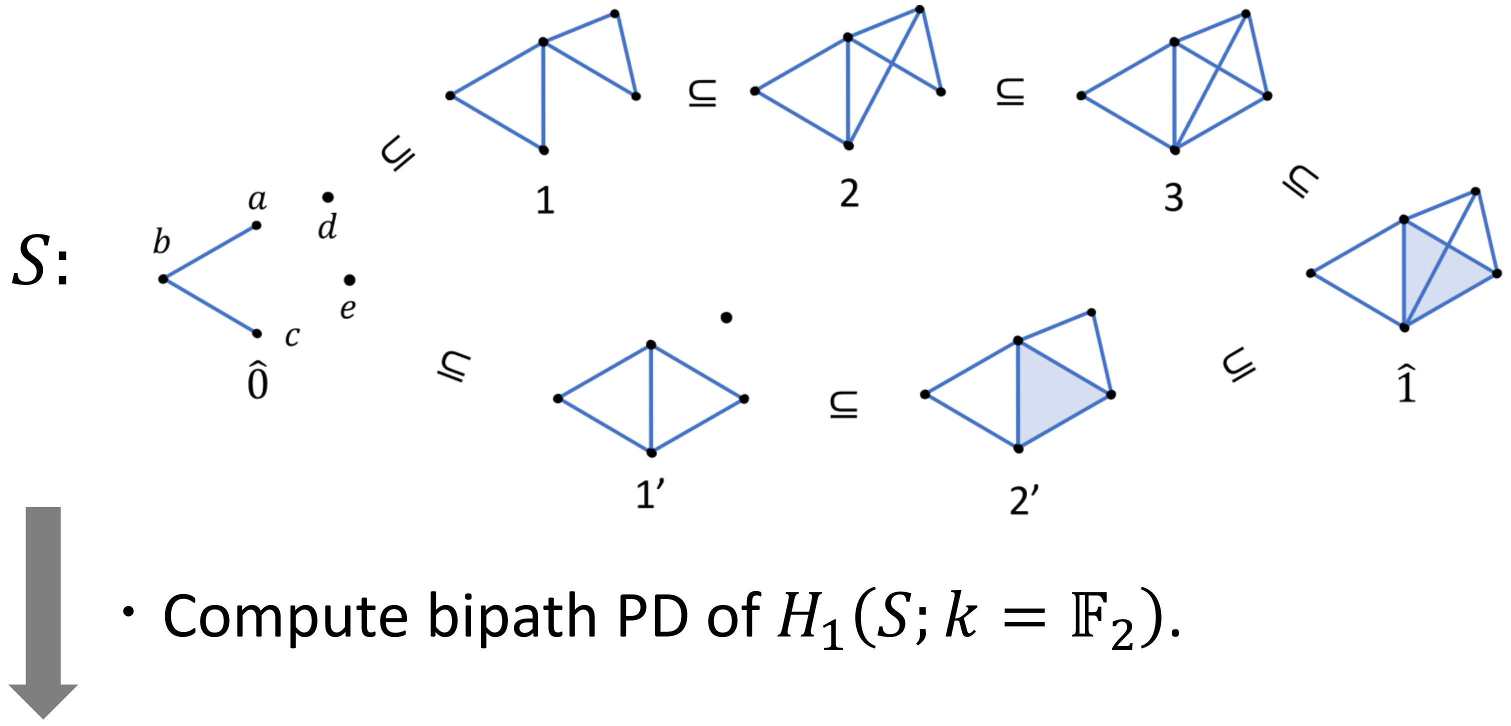
(2)Plot a point on the upper left region "$\mathcal{B}$" for the interval $B_{n,m}$.

(3)Plot a point (s, t) for the interval $\langle s, t \rangle$.



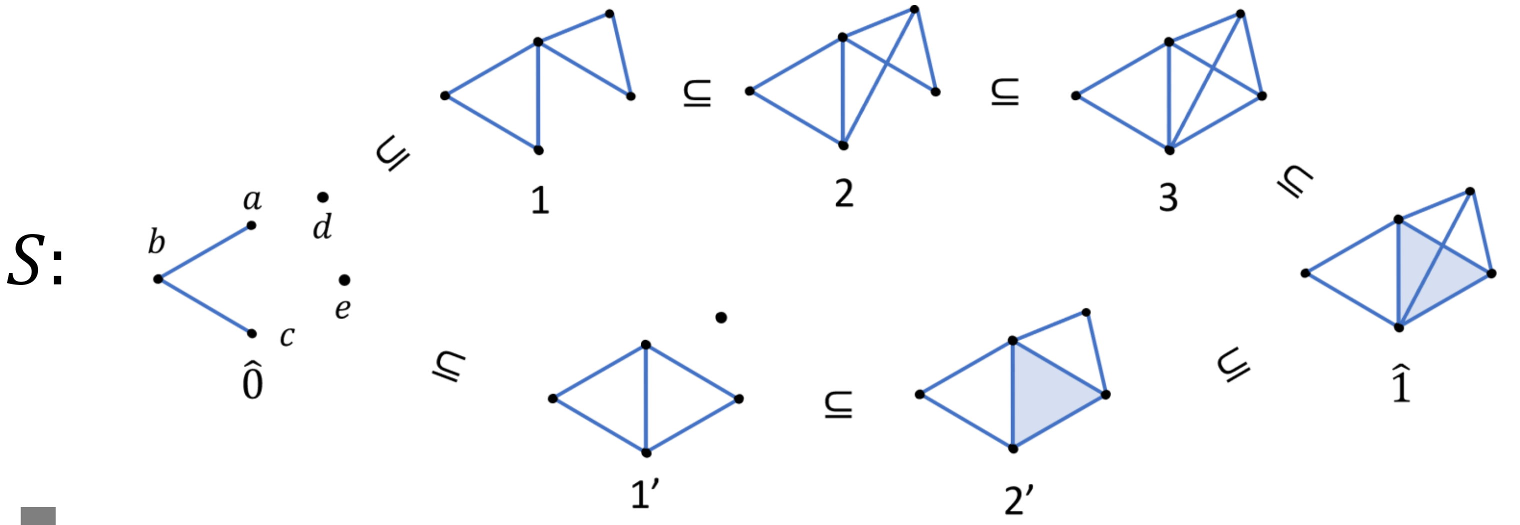
Bipath PD: multiset of points in the plane

Examples of bipath PD

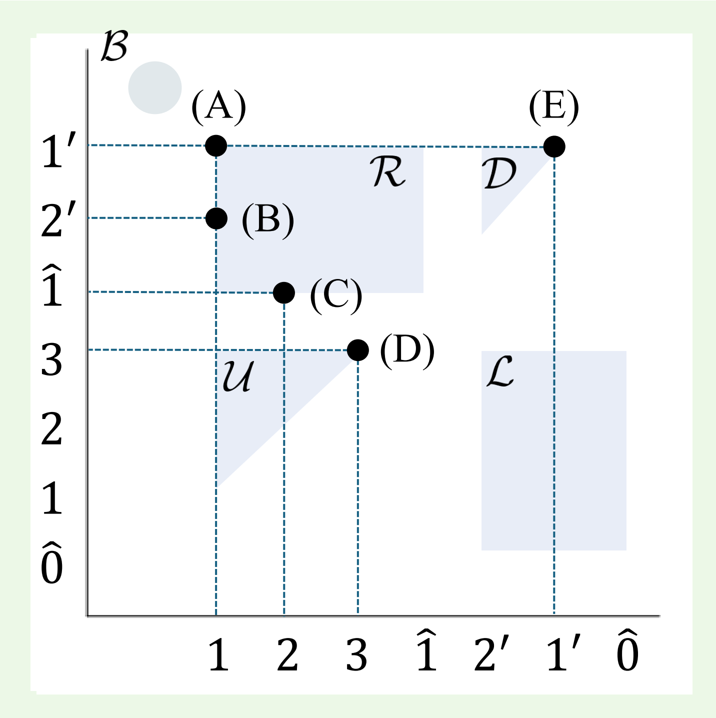


- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

Examples of bipath PD



1st.



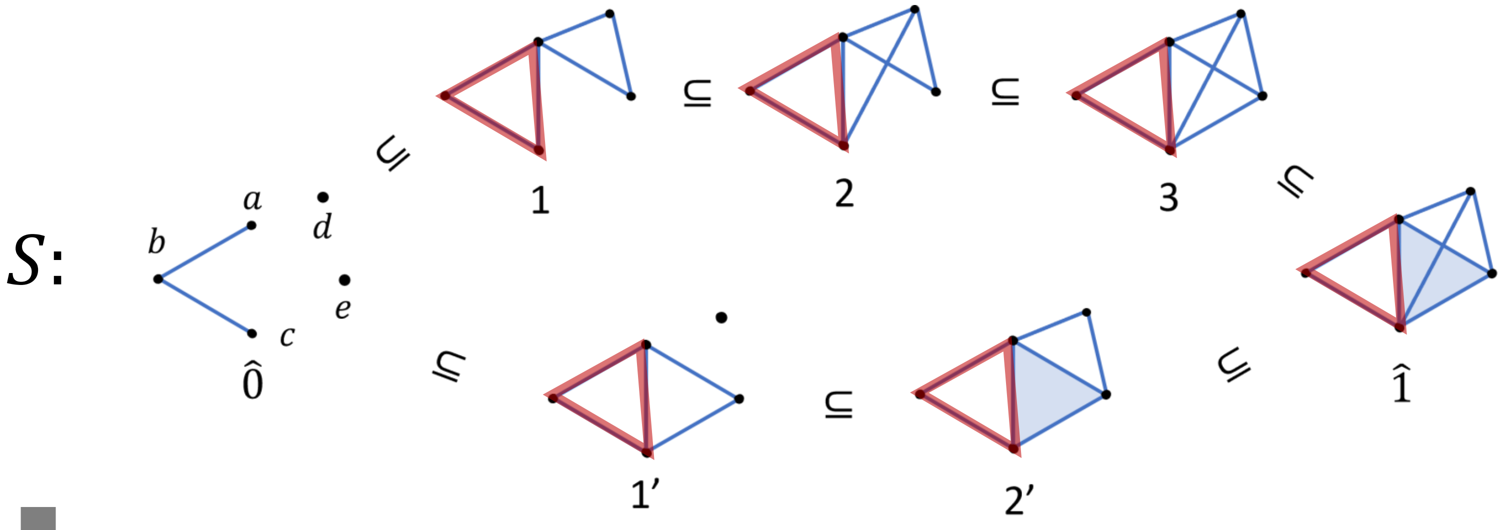
- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

$$\mathcal{B}(H_1(S;k)) = \{\{1,2,3,\hat{1},2',1'\},\{1,2,3,\hat{1},2'\},\{2,3,\hat{1}\},\{3\},\ \{1'\}\}$$

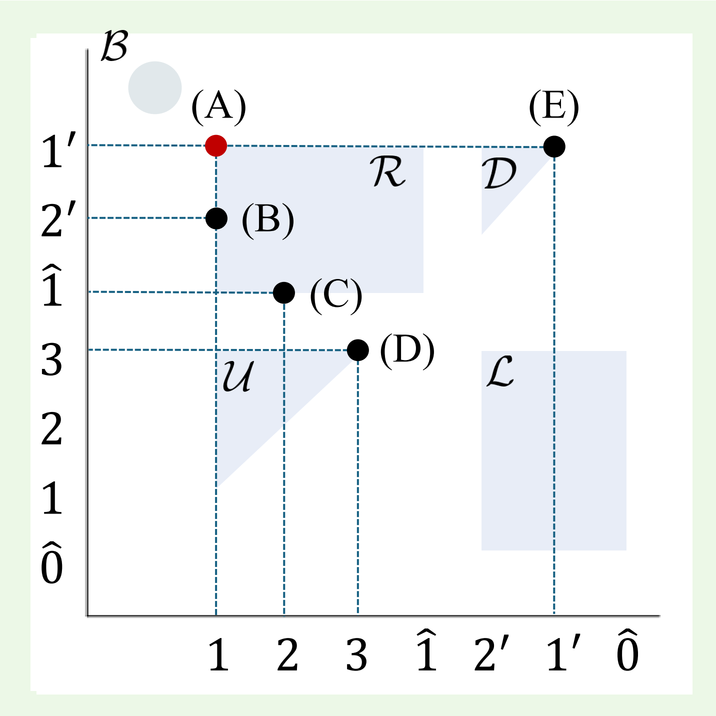
$$= \{\langle 1,1'\rangle, \langle 1,2'\rangle, \langle 2,\hat{1}\rangle, \langle 3,3\rangle, \langle 1',1'\rangle\}.$$

(A) (B) (C) (D) (E)

Examples of bipath PD



1st.



- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

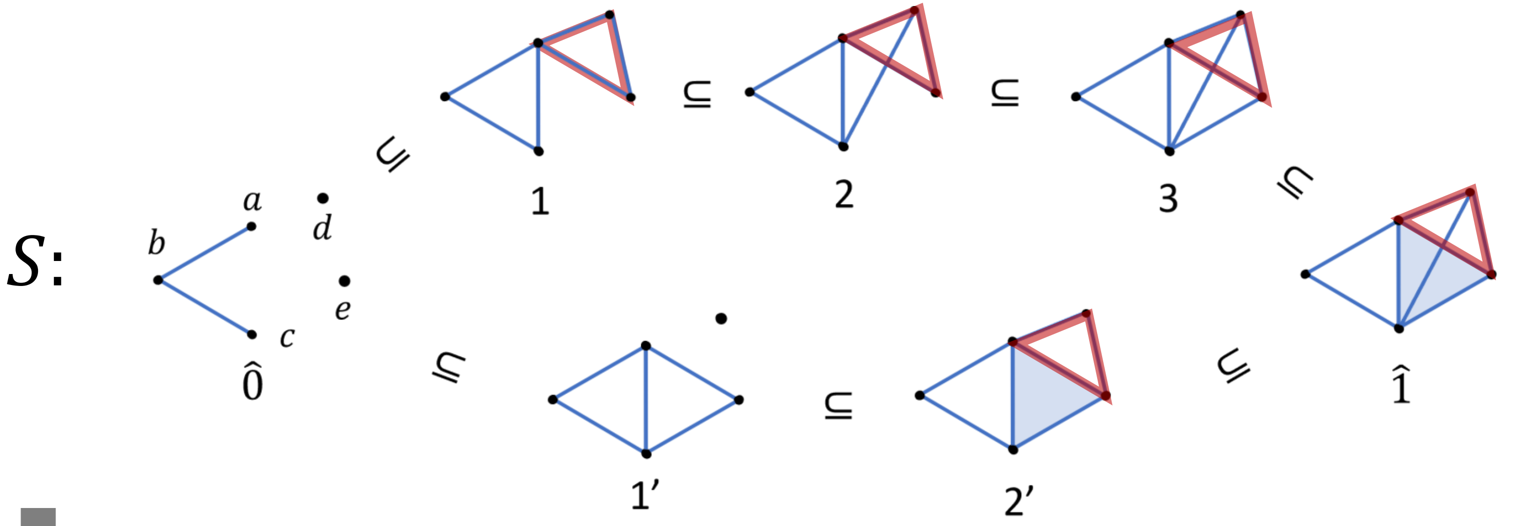
$$\mathcal{B}(H_1(S;k)) = \{\{1,2,3,\hat{1},2',1'\}, \{1,2,3,\hat{1},2'\}, \{2,3,\hat{1}\}, \{3\}, \{1'\}\}$$

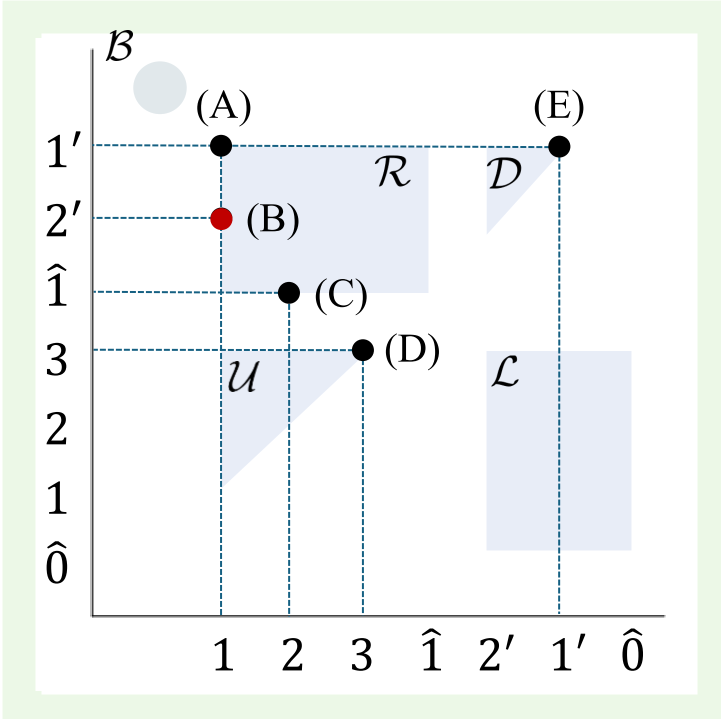
$$= \{\langle 1,1'\rangle, \langle 1,2'\rangle, \langle 2,\hat{1}\rangle, \langle 3,3\rangle, \langle 1',1'\rangle\}.$$

(A) (B) (C) (D) (E)

Examples of bipath PD

1st.





・Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

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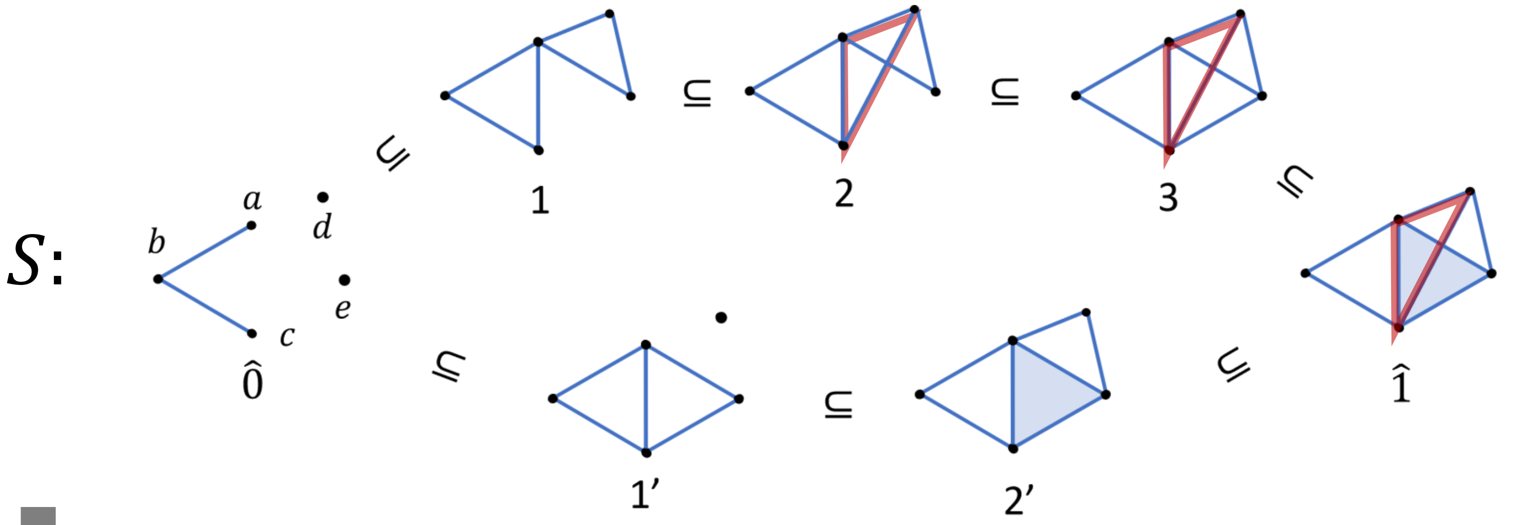
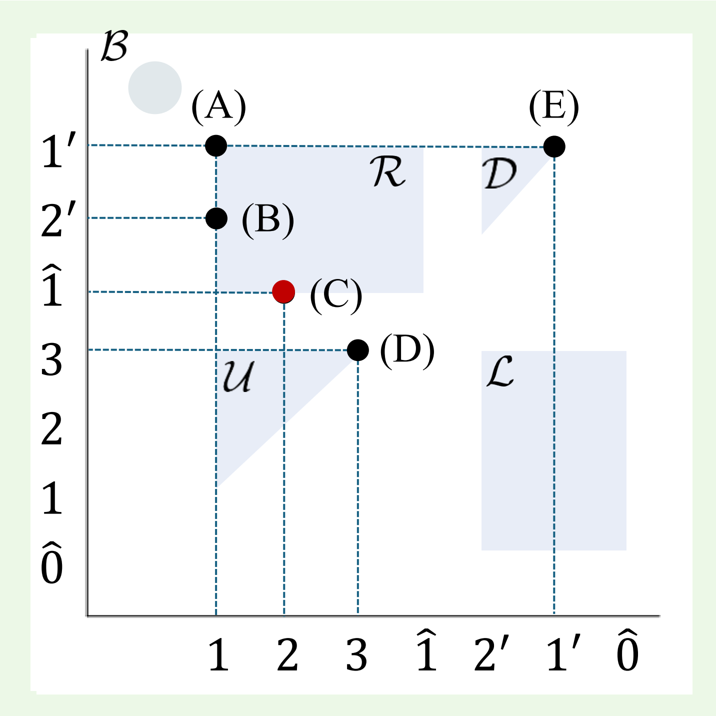
$$= \{\langle 1,1'\rangle, \langle 1,2'\rangle, \langle 2,\hat{1}\rangle, \langle 3,3\rangle, \langle 1',1'\rangle\}.$$

(A) (B) (C) (D) (E)

Examples of bipath PD

1st.





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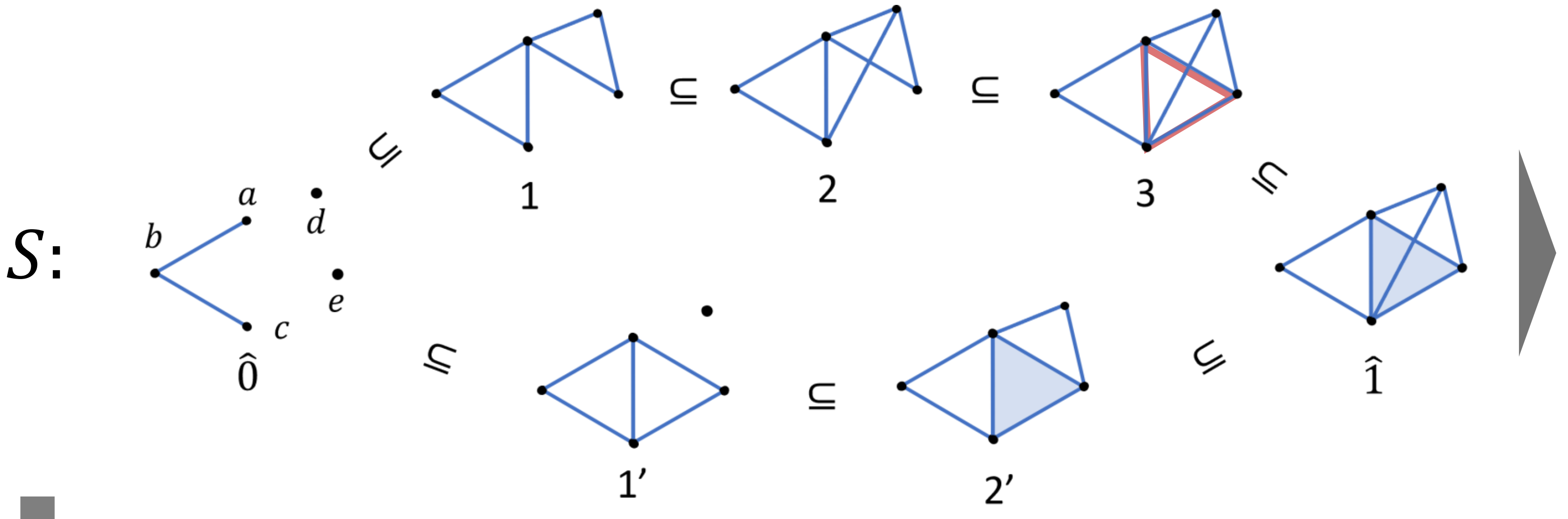
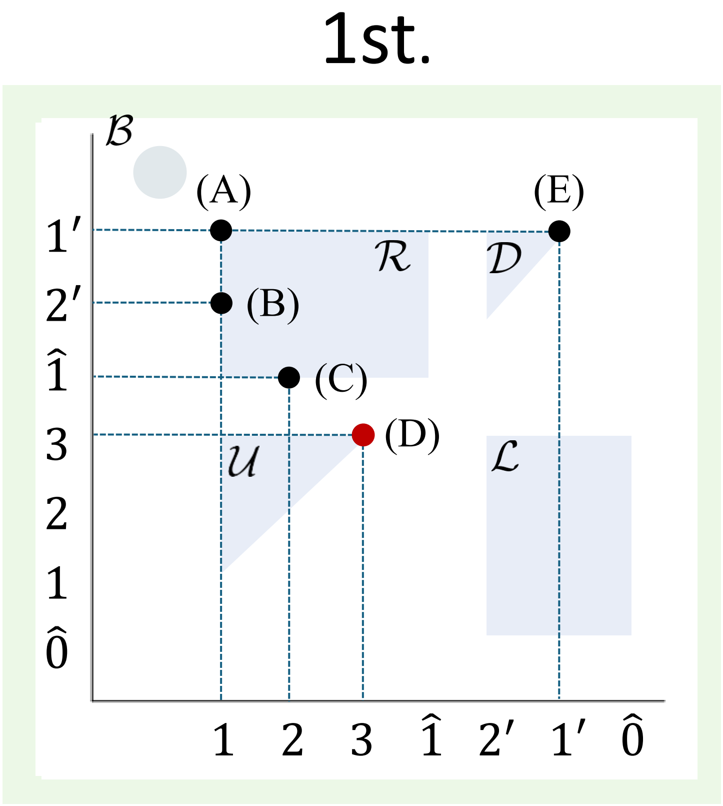
$$= \{\langle 1,1'\rangle, \langle 1,2'\rangle, \langle 2,\hat{1}\rangle, \langle 3,3\rangle, \langle 1',1'\rangle\}.$$

(A) (B) (C) (D) (E)

Examples of bipath PD





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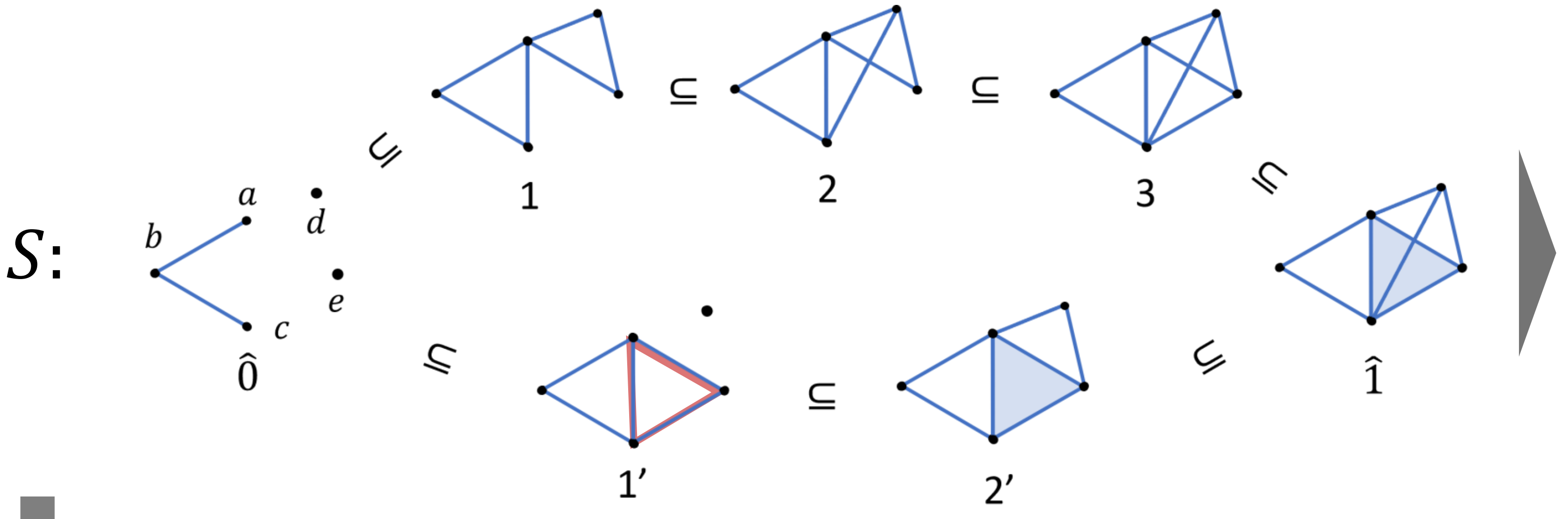
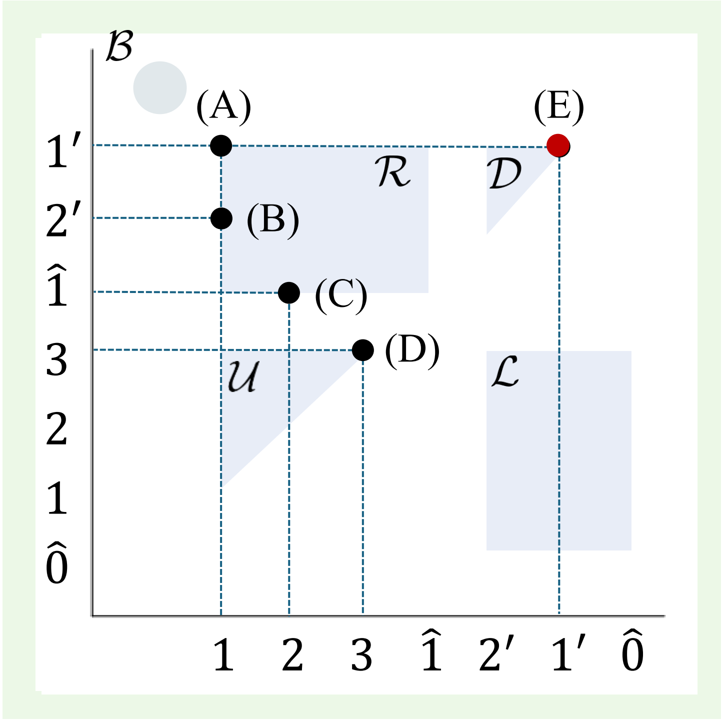
$$= \{\langle 1,1' \rangle, \langle 1,2' \rangle, \langle 2,\hat{1} \rangle, \langle 3,3 \rangle, \langle 1',1' \rangle\}.$$

(A) (B) (C) (D) (E)

Examples of bipath PD



1st.



- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.

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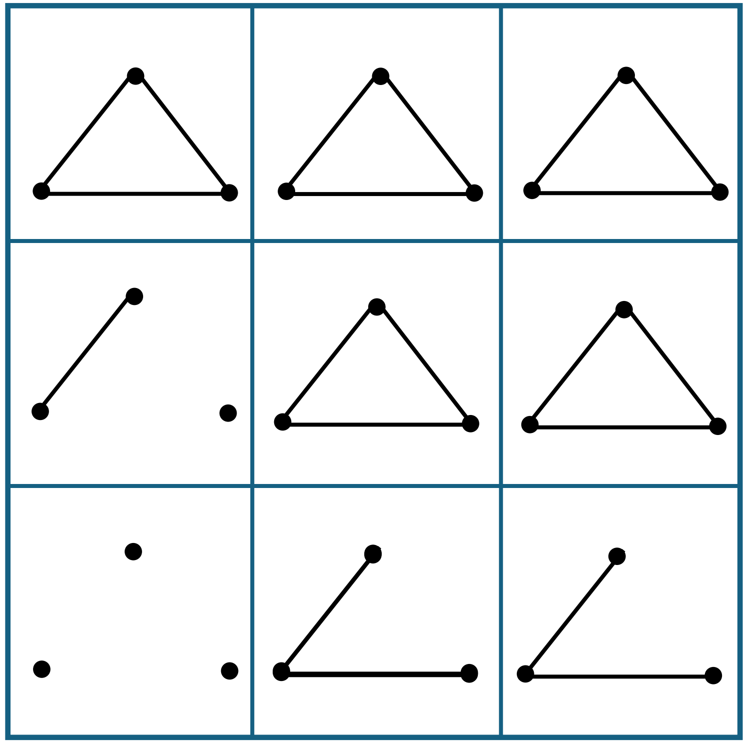
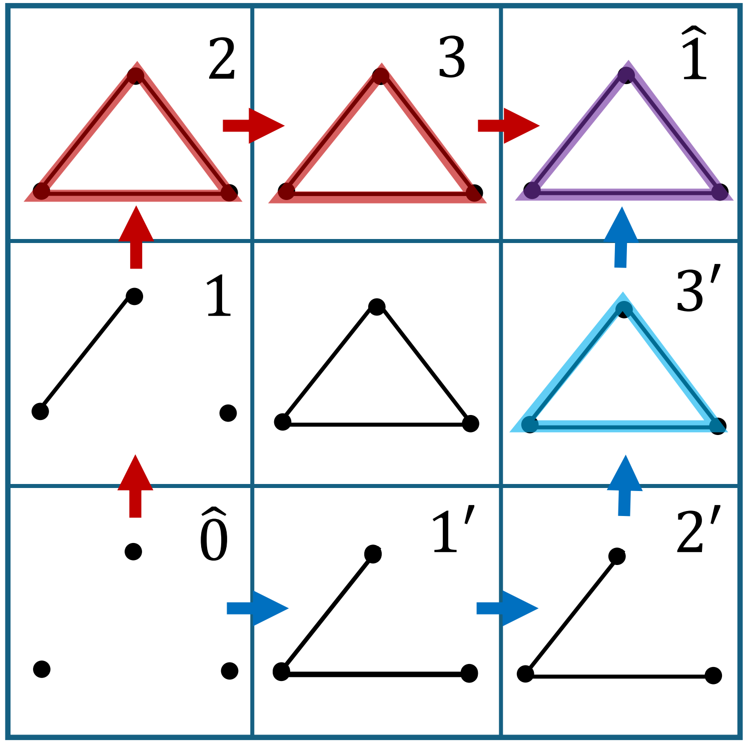
(A) (B) (C) (D) (E)

Examples of bipath PD



Examples of bipath PD

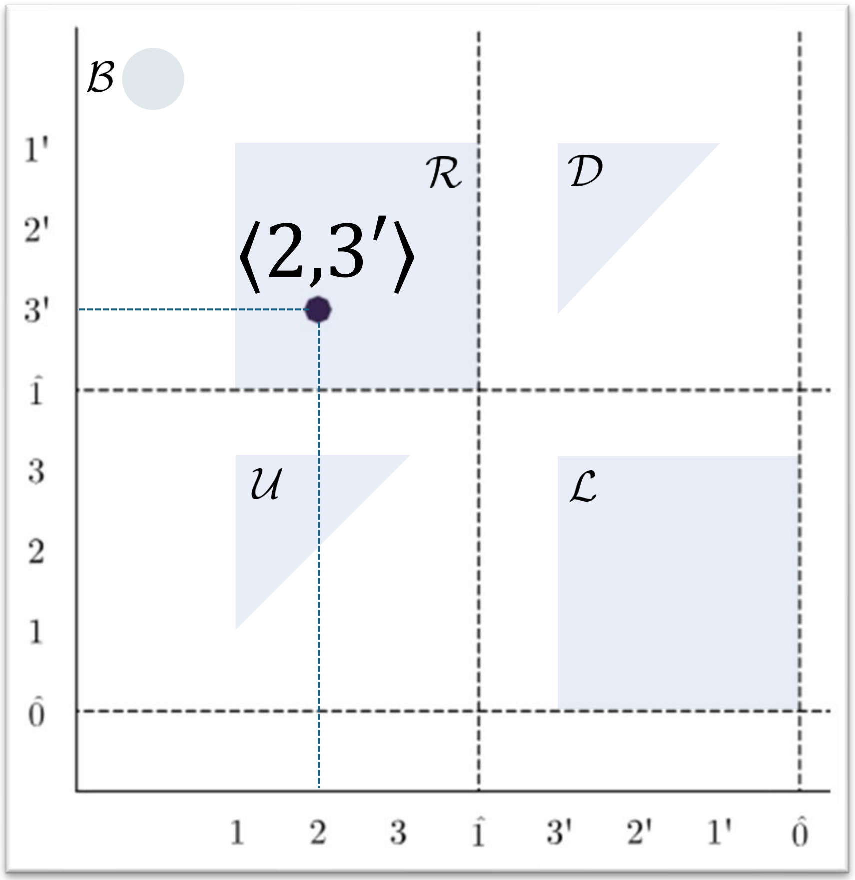


→ :Upper path

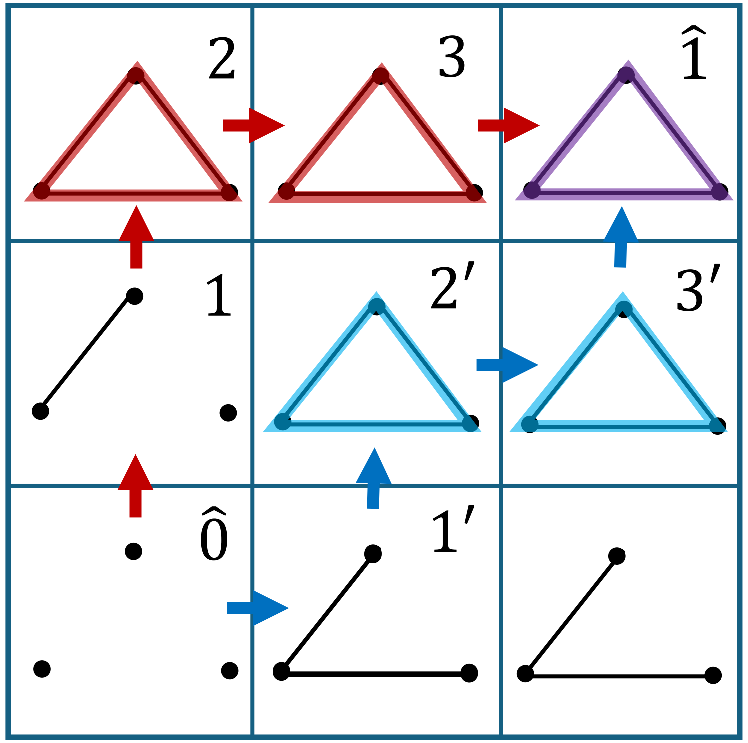
→ :Lower path

Get bipath PH.
$(k = \mathbb{F}_2)$

1st.



Examples of bipath PD

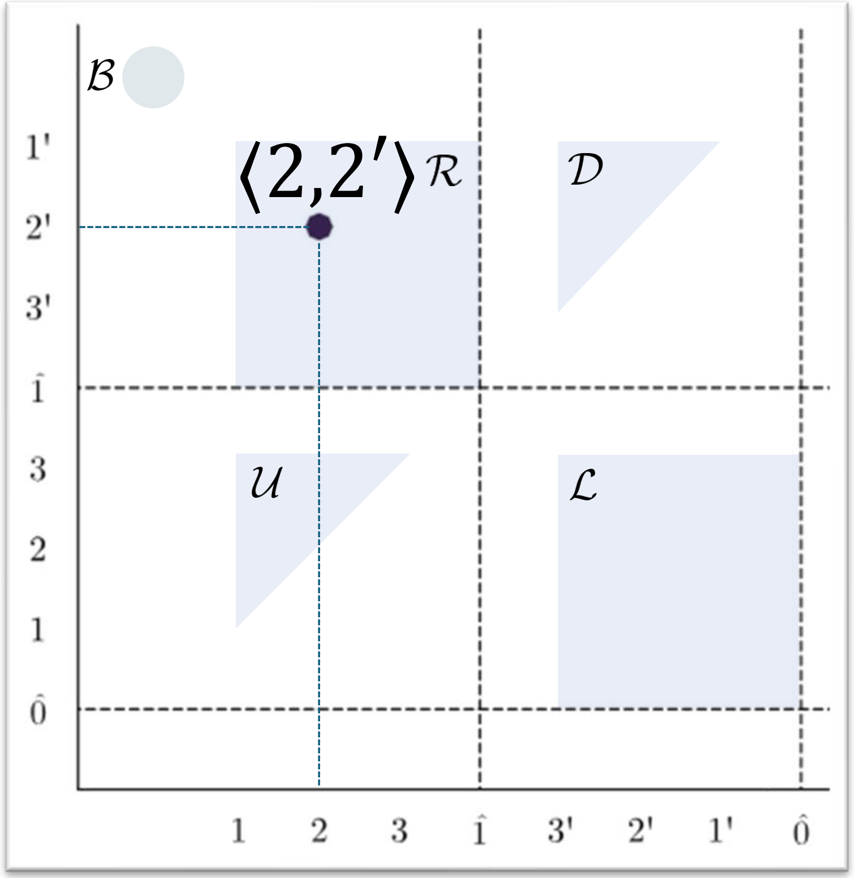


:Upper path

:Lower path

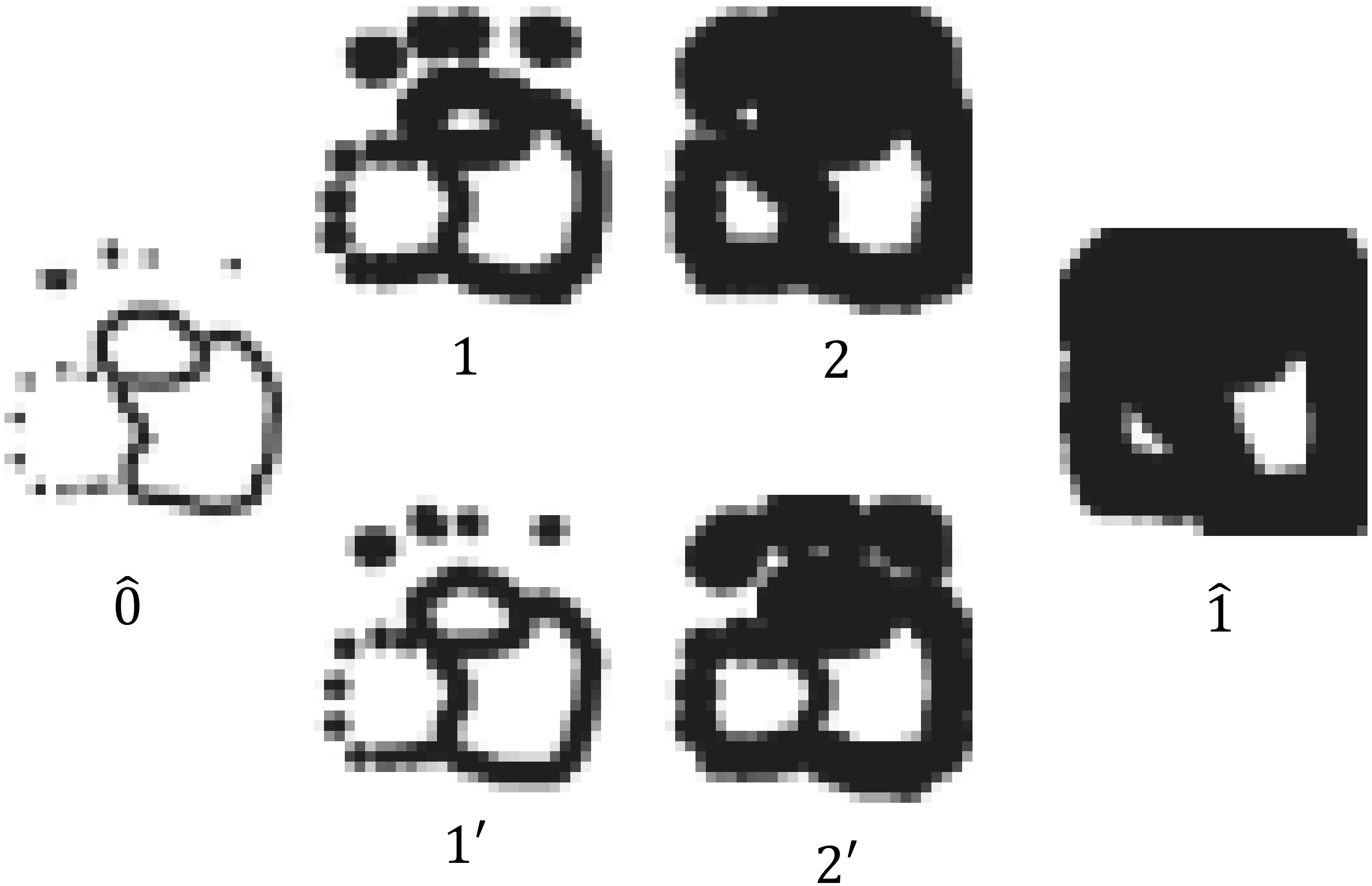
Get bipath PH.
$(k = \mathbb{F}_2)$

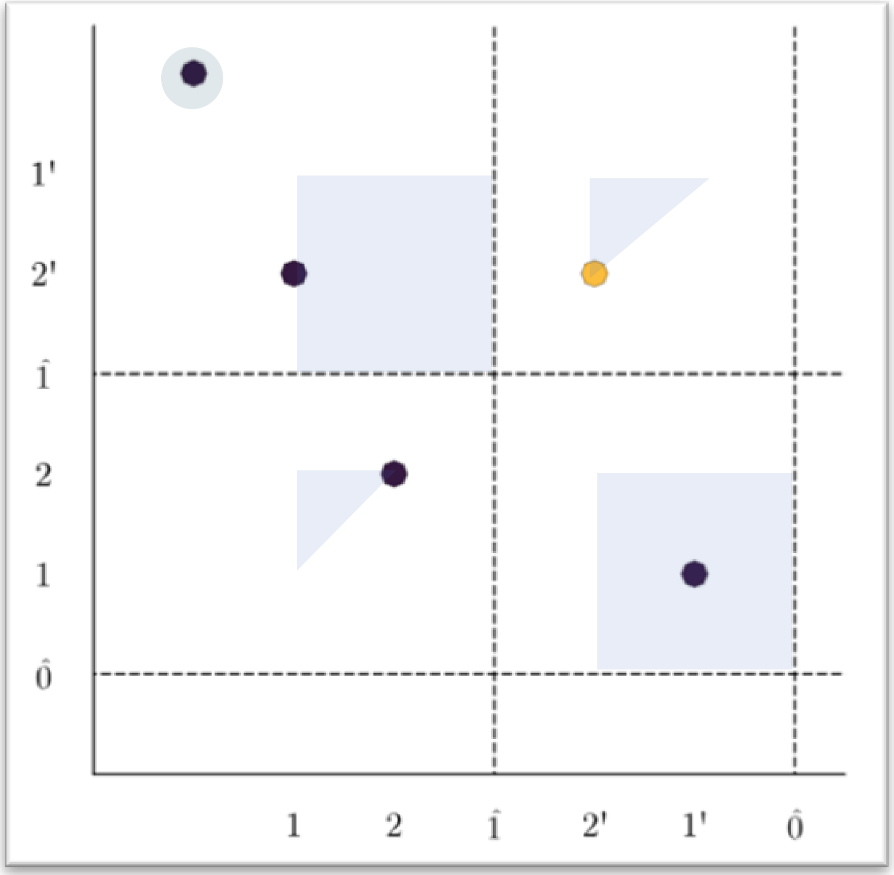
1st.



Examples of bipath PD

1st.

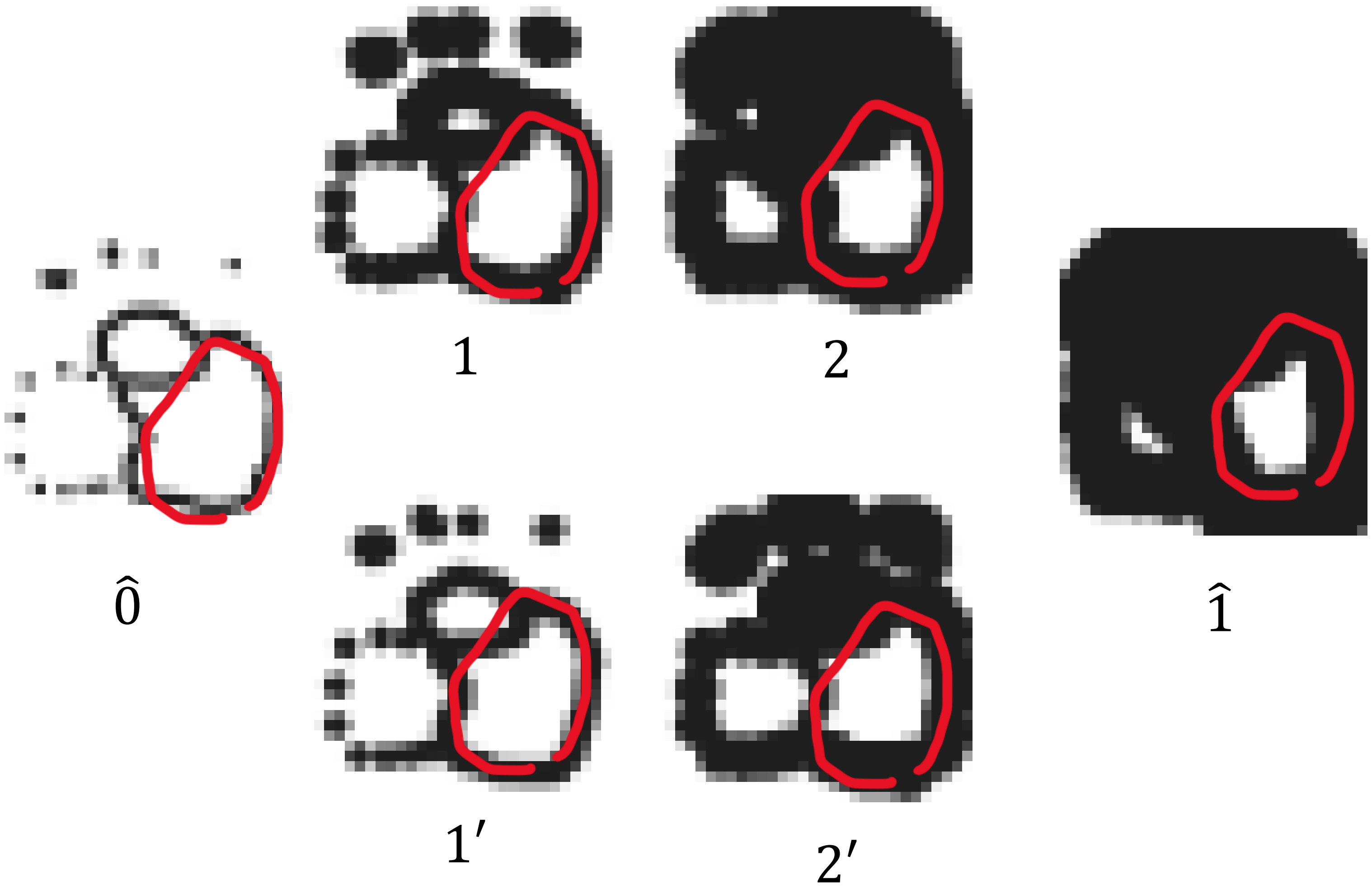




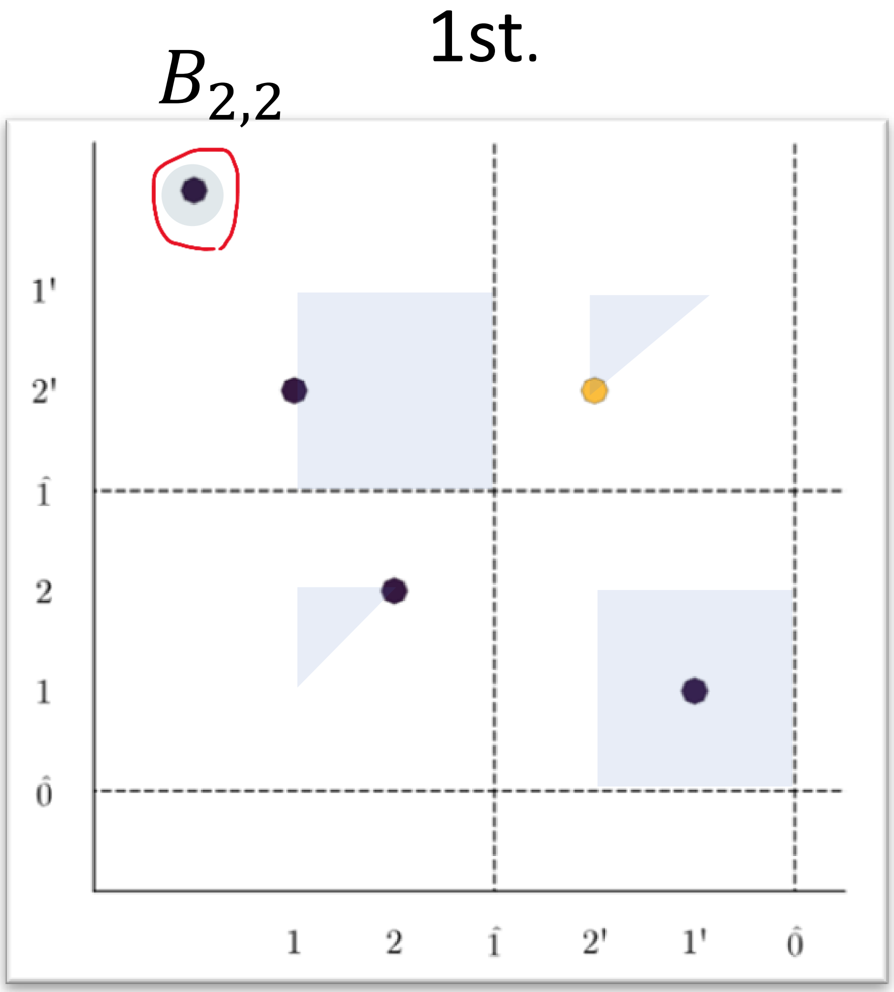
Bipath filtration of image data (30 × 30pixel)

Colors are changed by the multiplicity of intervals.

Examples of bipath PD

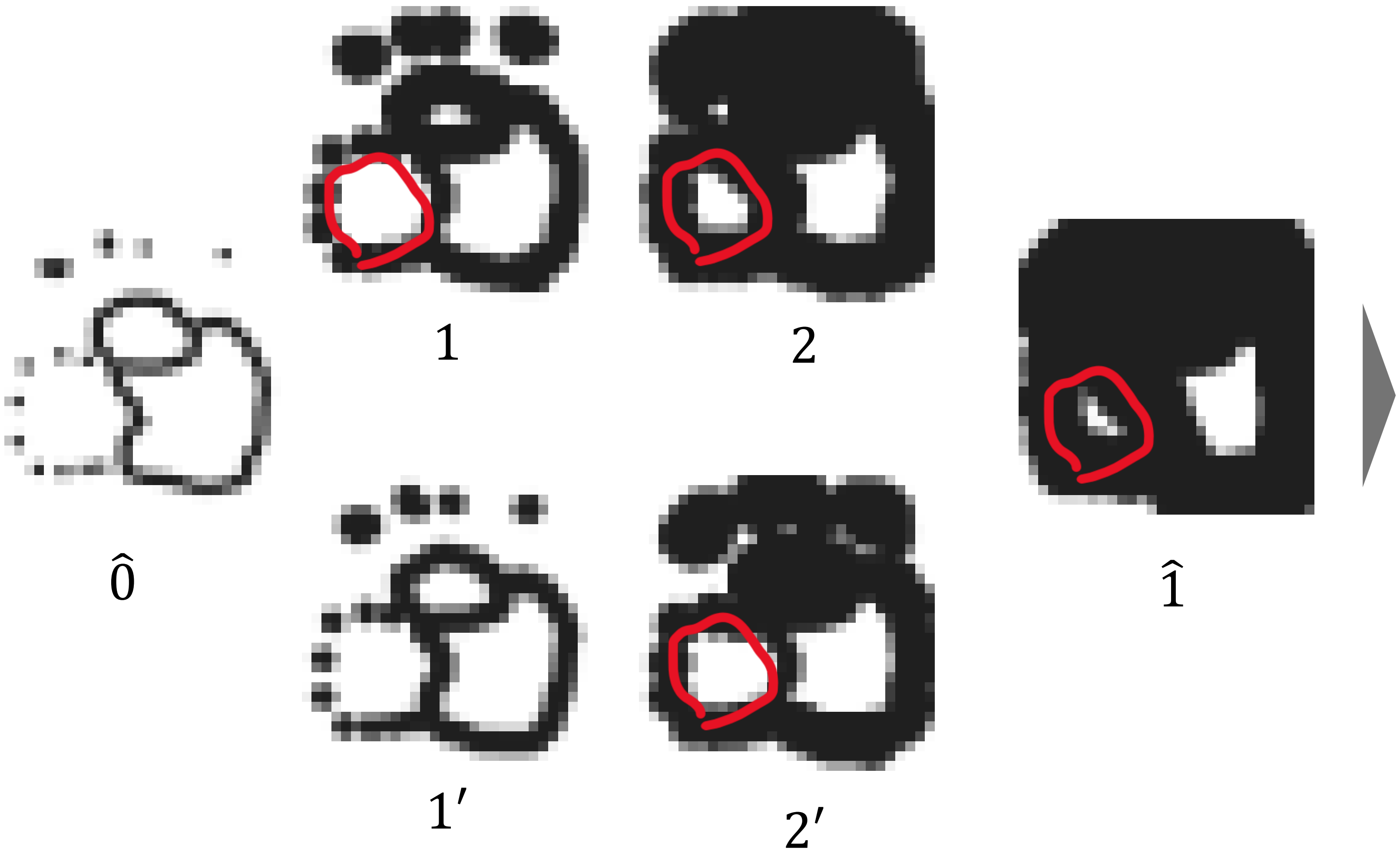


Bipath filtration of image data (30 × 30pixel)



Colors are changed by the multiplicity of intervals.

Examples of bipath PD



Bipath filtration of image data (30 × 30pixel)

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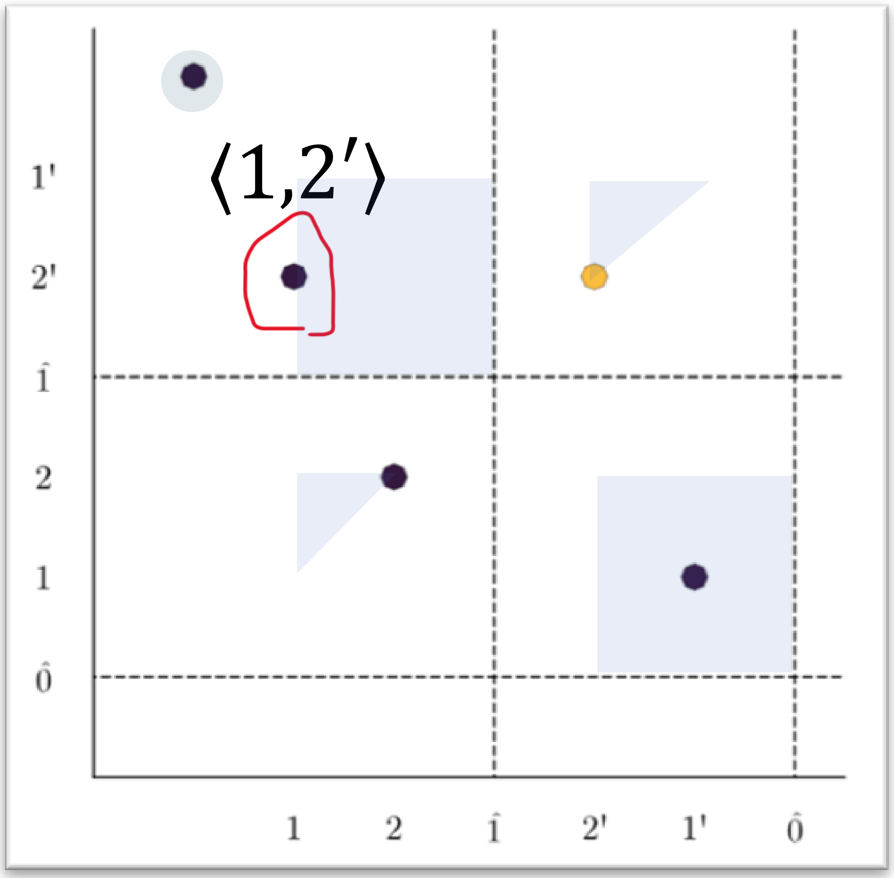
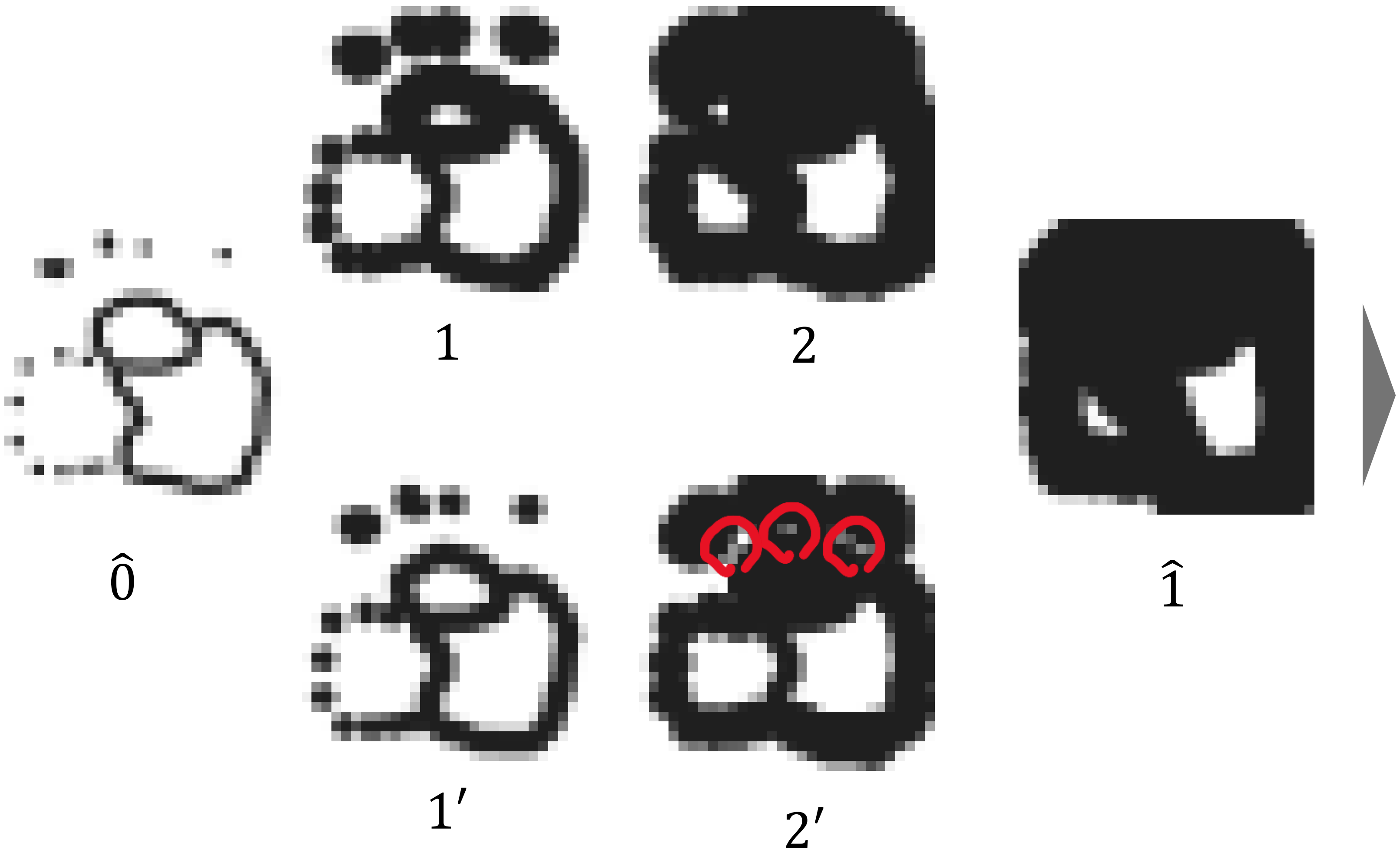


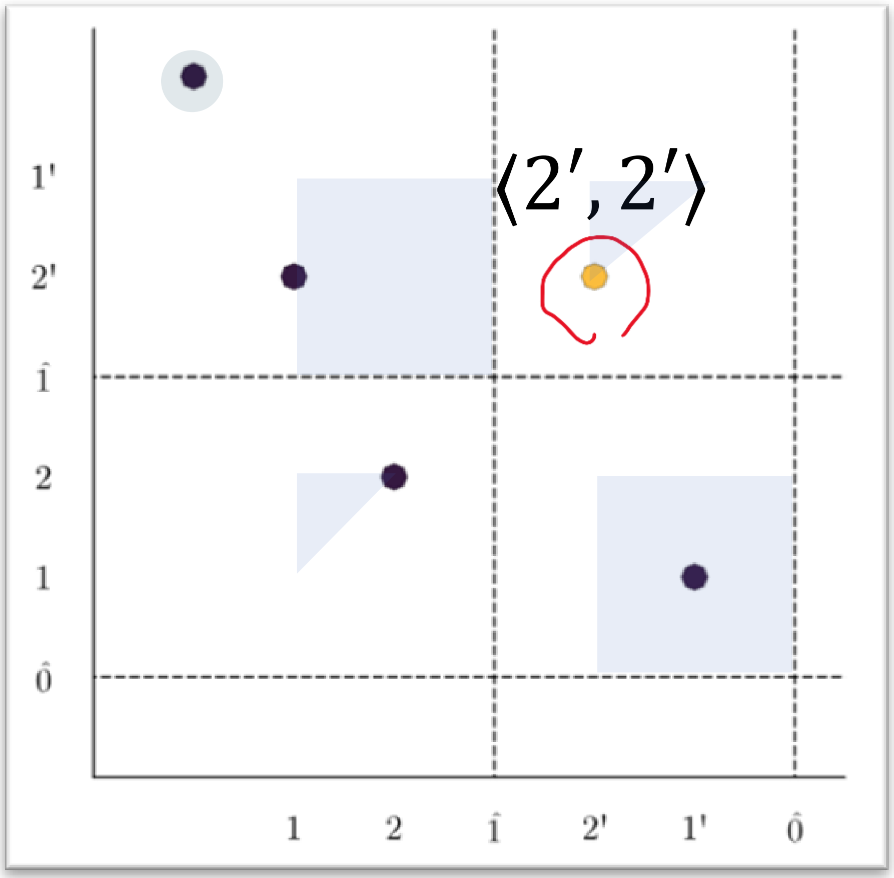
Colors are changed by the multiplicity of intervals.

Examples of bipath PD



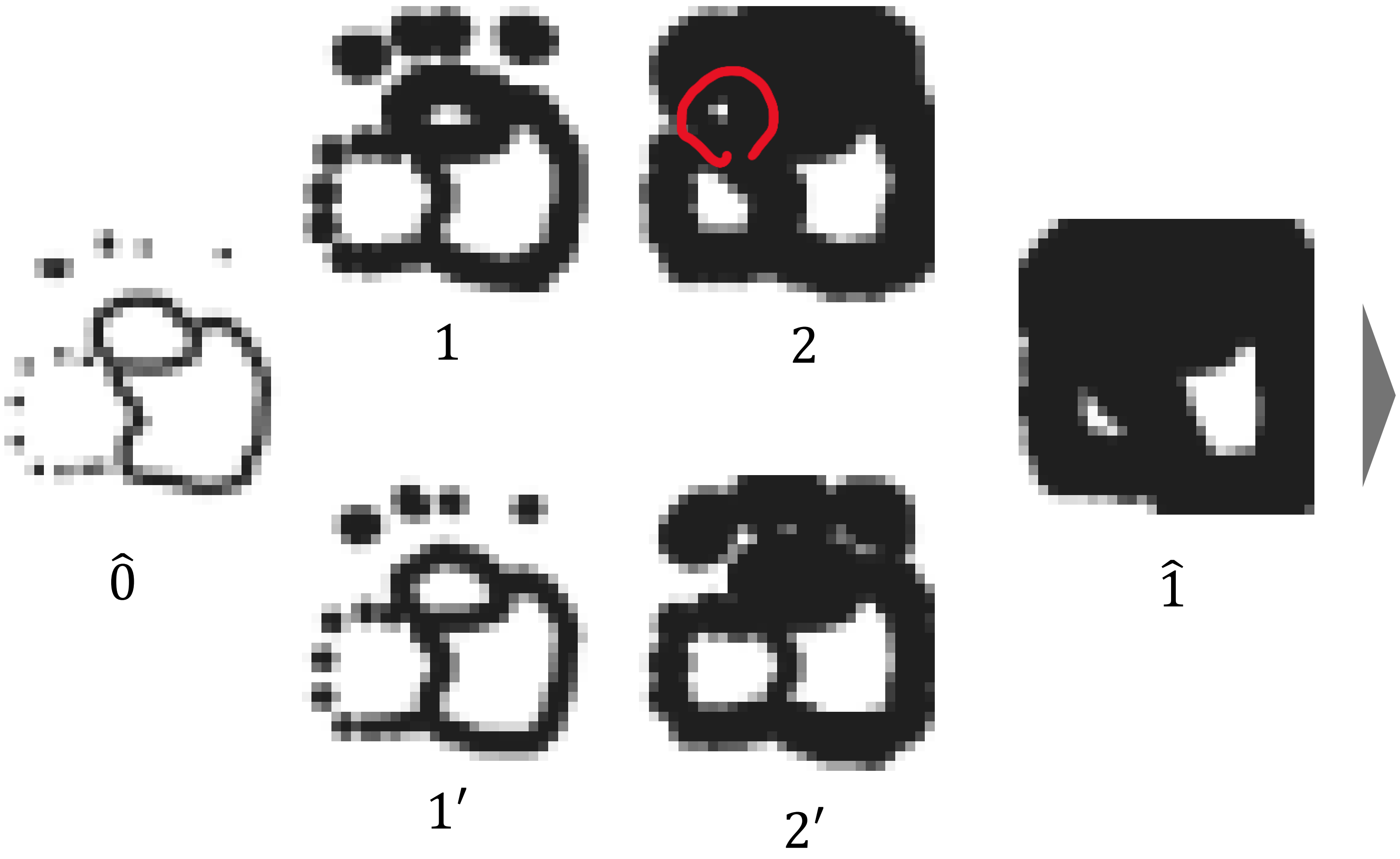
Bipath filtration of image data (30 × 30pixel)

1st.



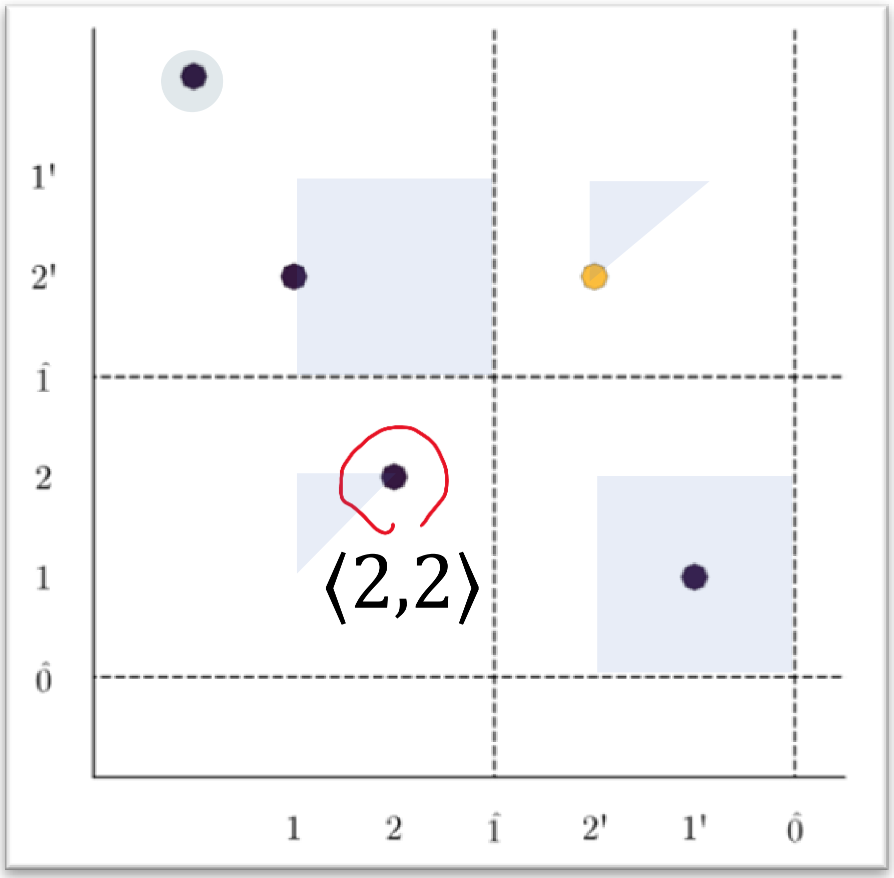
Colors are changed by the multiplicity of intervals.

Examples of bipath PD



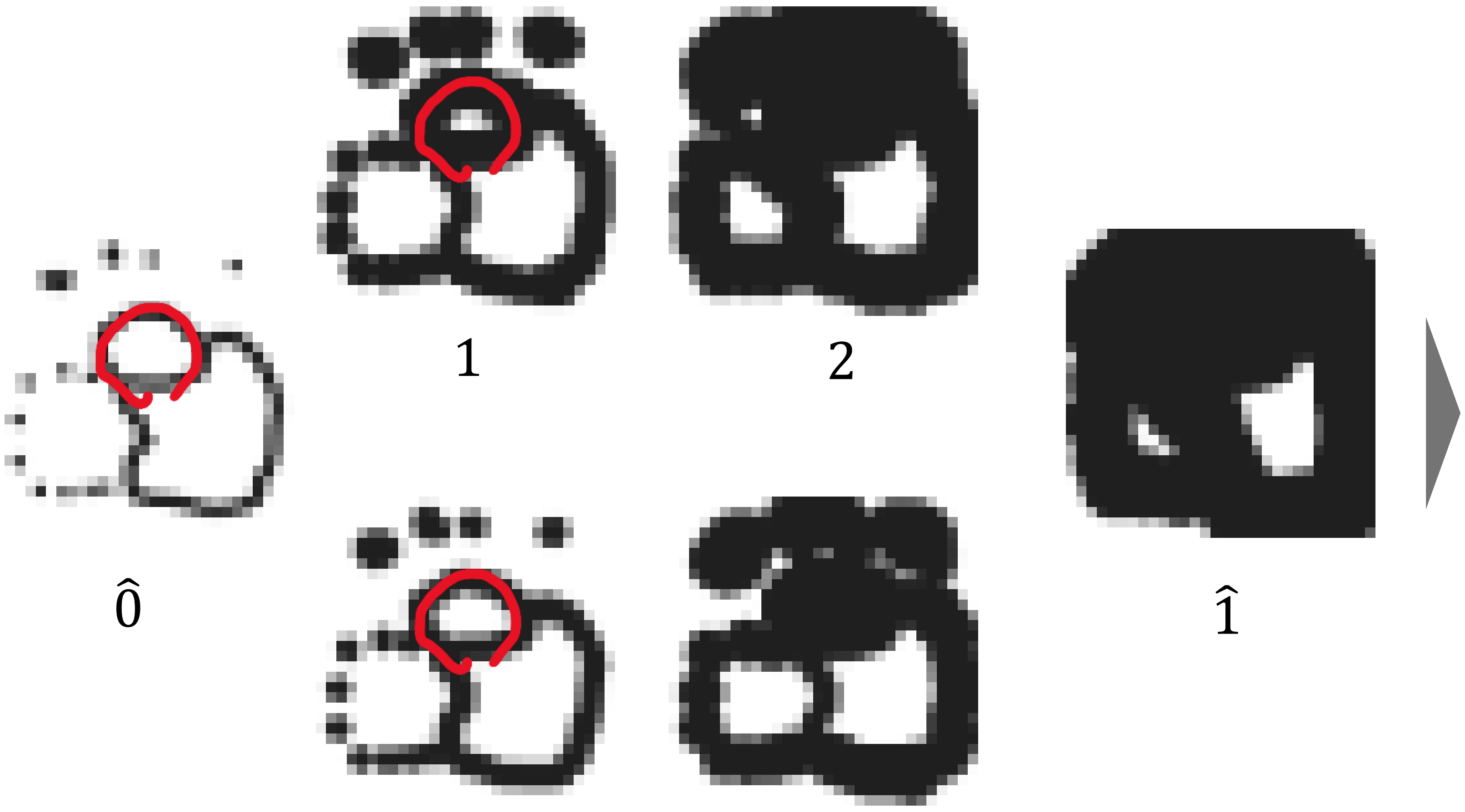
Bipath filtration of image data (30 × 30pixel)

1st.



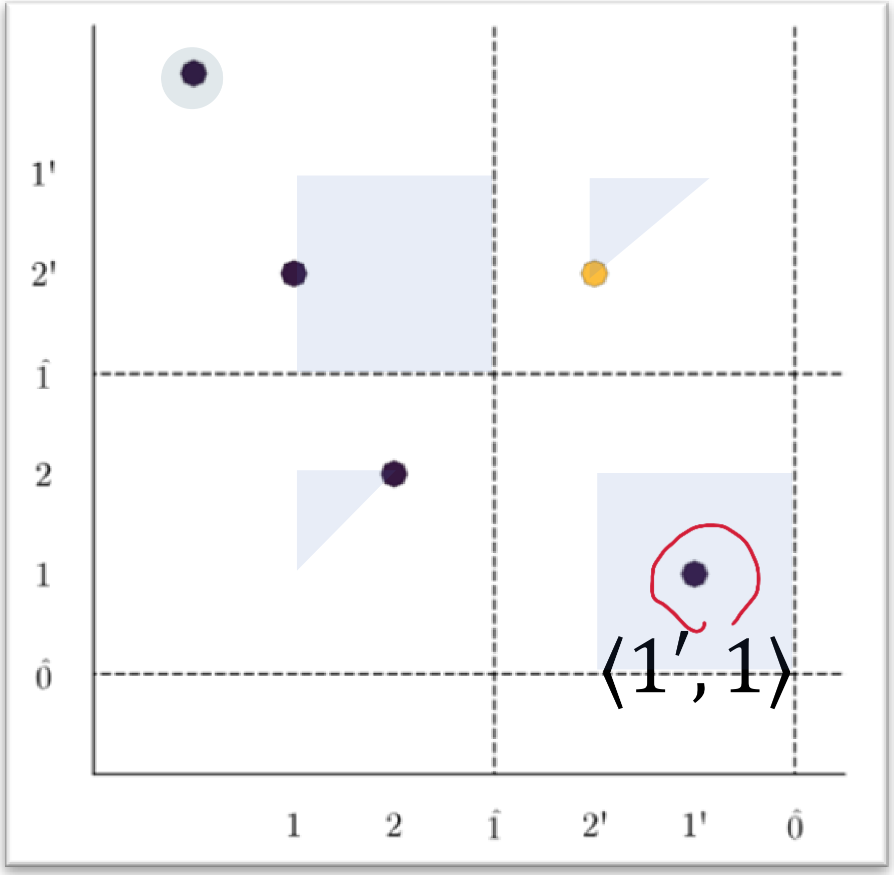
Colors are changed by the multiplicity of intervals.

Examples of bipath PD



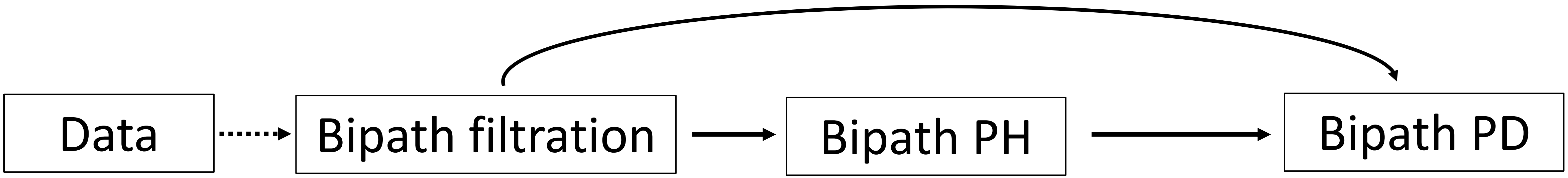
Bipath filtration of image data (30 × 30pixel)

1st.



Colors are changed by the multiplicity of intervals.

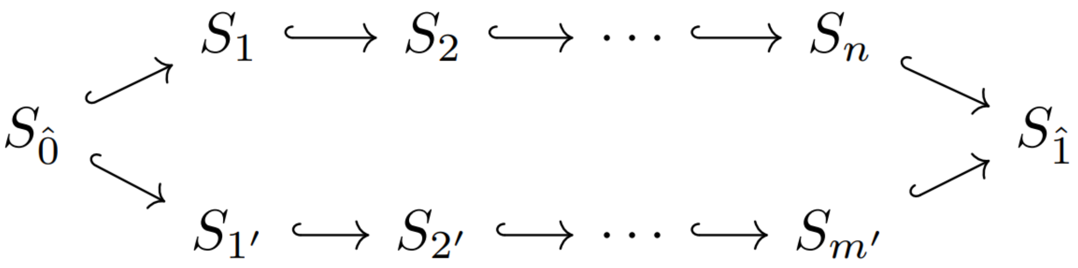
Implementation: Computing Bipath PD



Implementation: Computing bipath PD

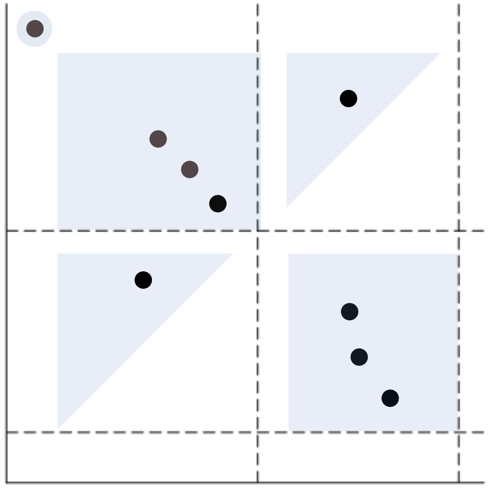
We gave a software for computing bipath PD on GitHub.
(https://github.com/ShunsukeTada1357/Bipathposets)

Input



Bipath filtration
(of simplicial complex)

Output



Bipath PD

Remark.

The computational algorithm is given in [Aoki-Escolar-T, Algorithm 2, 25].

Implementation: Computing bipath PD

Note.

・We get a bipath PD by 2 times of standard algorithm for PH and matrix operations (whose size depend on the number of intervals).

Standard algorithm

$$S_{\hat{0}} \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{\hat{1}}$$

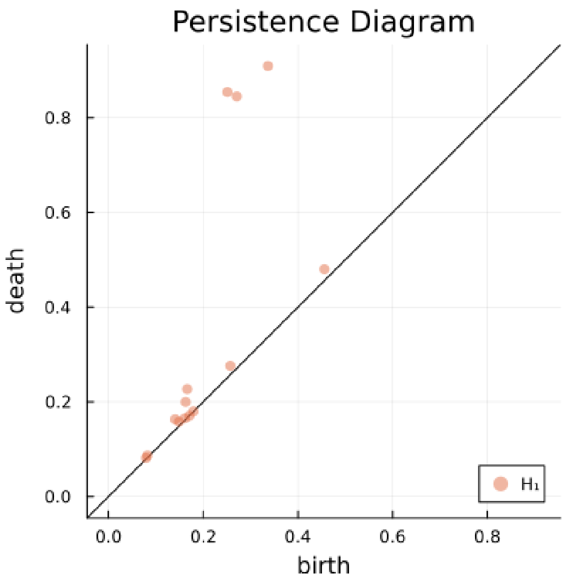
$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$S_{\hat{0}} \hookrightarrow S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} \hookrightarrow S_{\hat{1}}$$

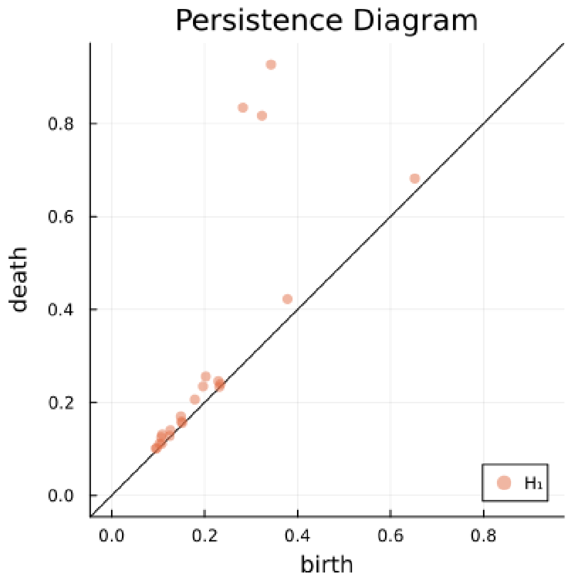
->Bipath PD can be computed without much more effort than the standard algorithm for PH.

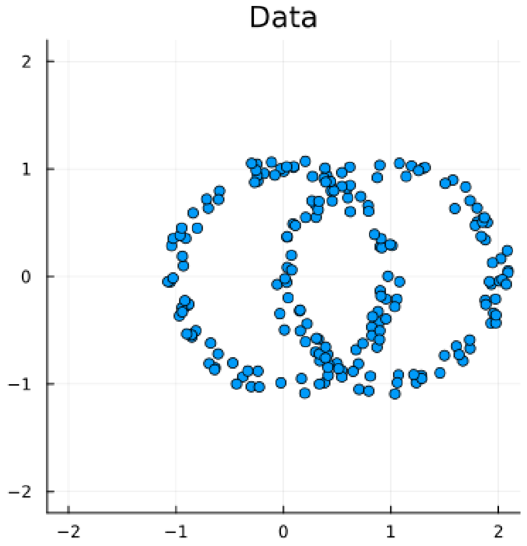
Stability of bipath persistence diagrams

Background: Stability theorem for standard PH

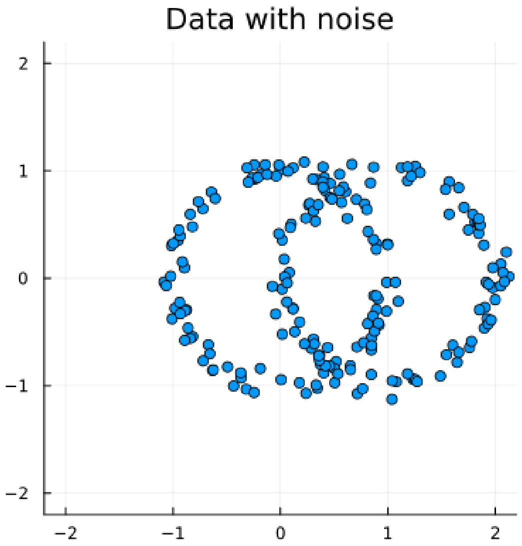


Small change





Small noise



Background: Stability theorem for standard PH

Stability theorem (see [Frédéric Chazal, et al., 2009] for example)

Let f and g be a real-valued function on top. sp. X. Then, we have

$$d_{\mathrm{B}}(\mathcal{B}(V_f), \mathcal{B}(V_g)) \leq ||f - g||_\infty.$$

- $\mathcal{B}(V_f)$: The standard PD of PH by sublevelset filtration of f.
- $\mathcal{B}(V_g)$: The standard PD of PH by sublevelset filtration of g.
- d_{B}: Bottleneck distance between PDs.
- $||f - g||_\infty := \sup_{x \in X} |f(x) - g(x)|$.

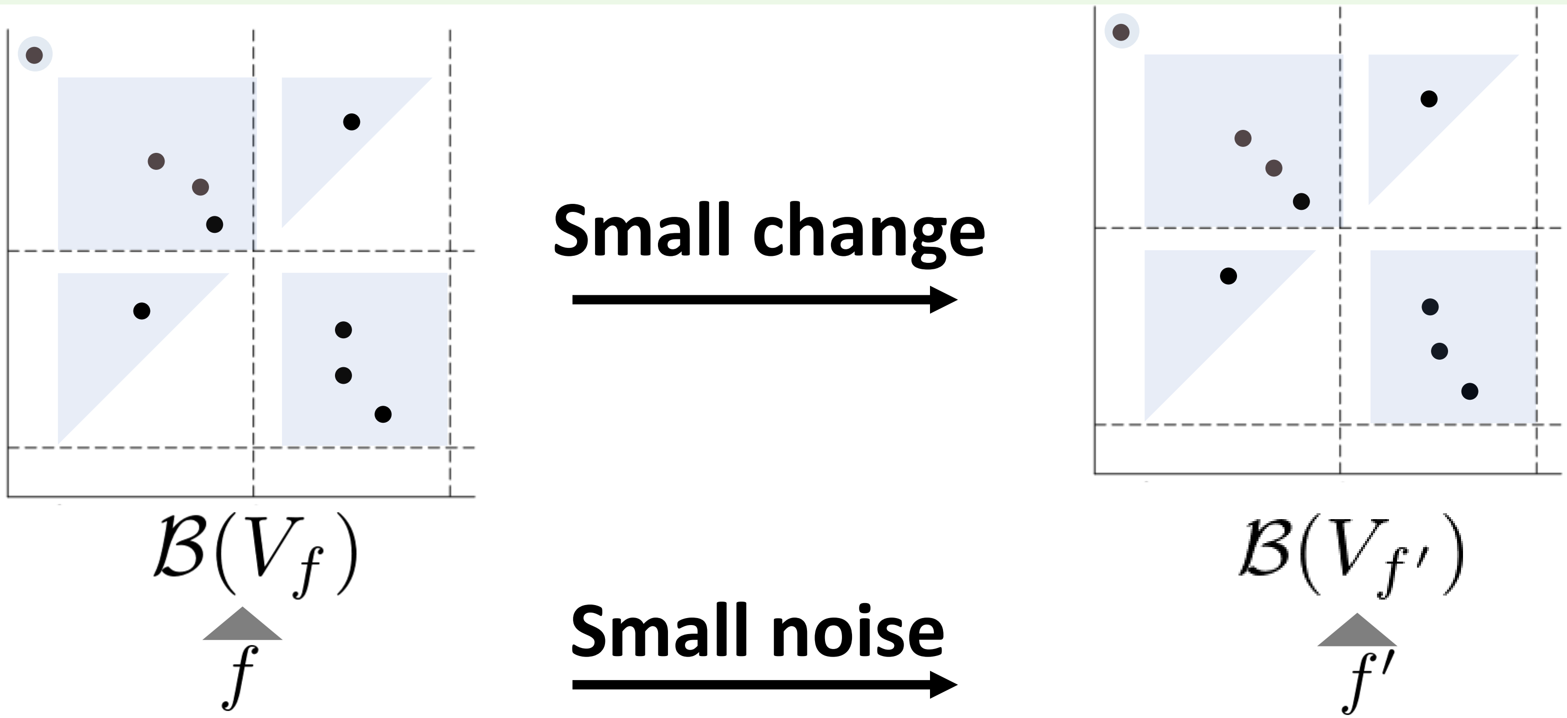
It gives mathematical justification of the use of PH for studying noisy data.

・David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. Discrete& Computational Geometry, 37:103–120, 2007.
・Frédéric Chazal, David Cohen-Steiner, Marc Glisse, Leonidas J Guibas, and Steve Y Oudot. Proximityof persistence modules and their diagrams. In Proceedings of the twenty-fifth annual symposium onComputational geometry, pages 237–246, 2009.

Stability theorem for bipath PD

Theorem [T, 25](Stability theorem for bipath PD).
Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be bipath functions satisfying (◆).
Then, we have the following inequality:

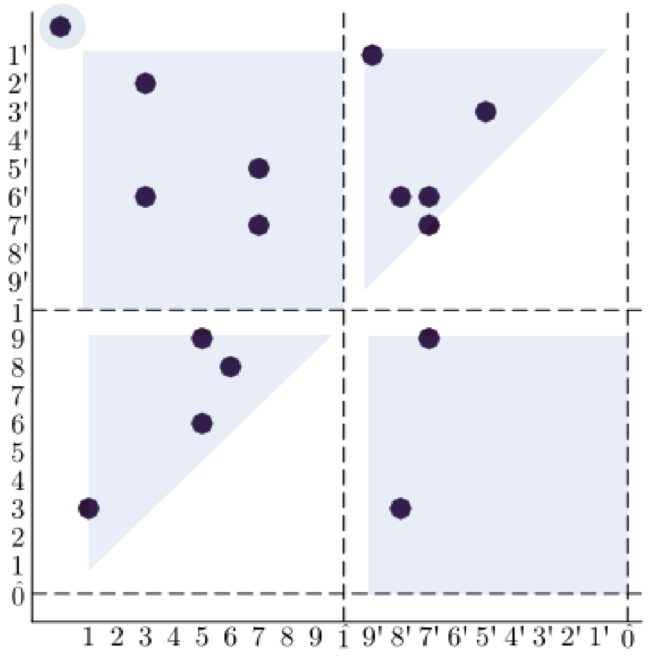
$$d_{\mathrm{B}}(\mathcal{B}(V_f), \mathcal{B}(V_g)) \leq ||f - g||_\infty.$$



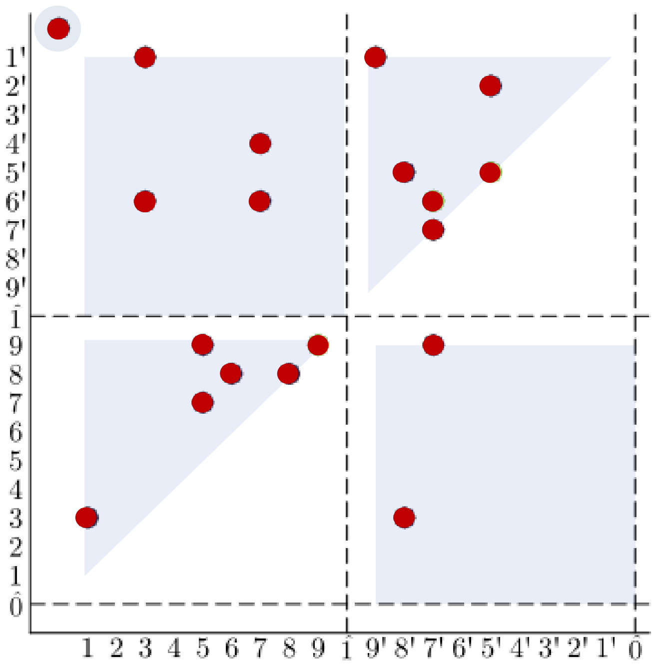
Stability theorem for bipath PD: Example using implementation

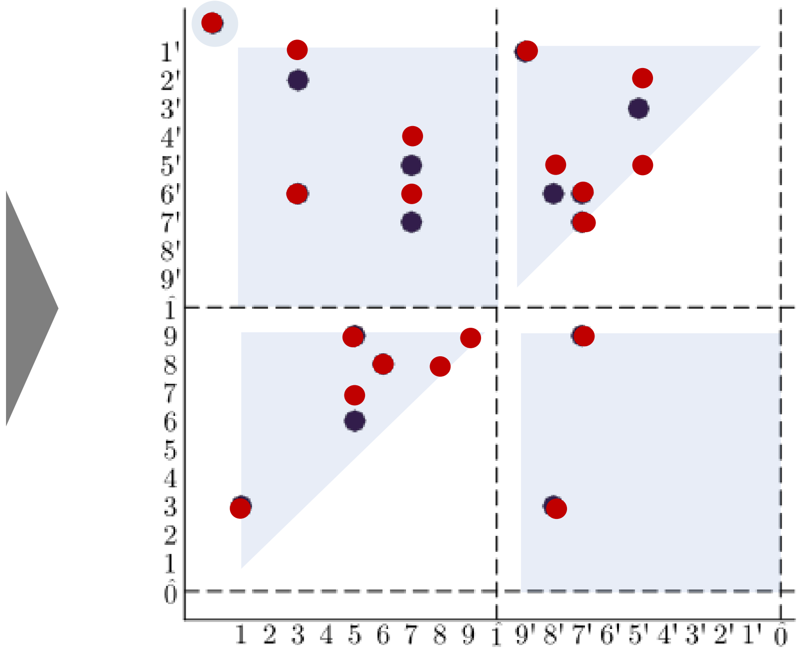
- f : bipath function of a simplicial complex.
- $f' = f + \text{noise}$ such that $\|f - f'\|_\infty = 1$ with (◆).

$f \Rightarrow \mathcal{B}(V_f)$



$f' \Rightarrow \mathcal{B}(V_{f'})$





overlay PDs

Stability theorem for bipath PD: Setting

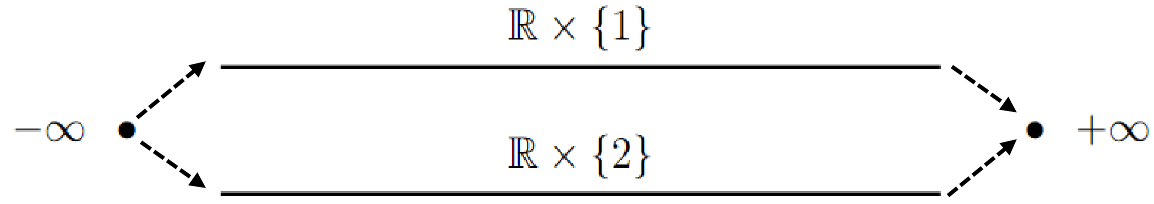
We consider continuous version of a bipath poset B to discuss stability.
(We have a natural embedding $\iota\colon B_{n,m} \hookrightarrow B$)

$B_{n,m}$:

$$\begin{array}{ccccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

$$\{1, \dots, n\} \sqcup \{1', \dots, m'\} \sqcup \{\hat{0}, \hat{1}\}$$

B:



$$(\mathbb{R} \times \{1\}) \sqcup (\mathbb{R} \times \{2\}) \sqcup \{-\infty, +\infty\}$$

Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath function)

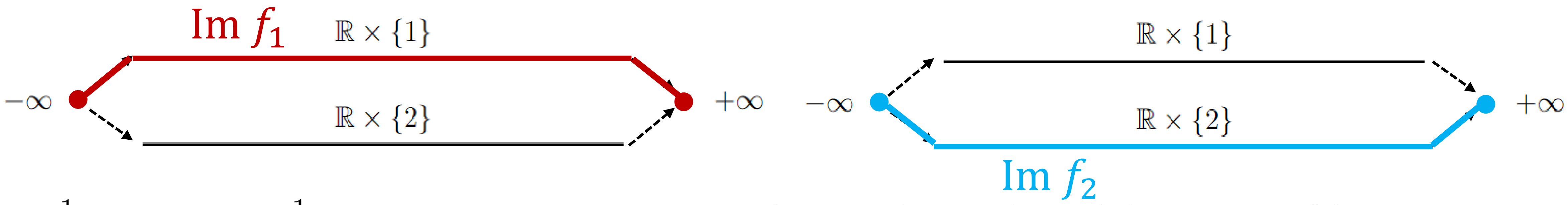
A *bipath function* f on top. sp. X is a pair of B-valued functions f_1 and f_2 on X such that

$$\mathrm{Im} f_i \subseteq (\mathbb{R} \times \{i\}) \sqcup \{\pm\infty\} \subseteq B$$

and

$$f_1^{-1}(-\infty) = f_2^{-1}(-\infty).$$

We denote by $f : X \to B$ a bipath function on X.



* $f_1^{-1}(-\infty) = f_2^{-1}(-\infty)$ is needed to define a bipath sublevelset filtration.

Stability theorem for bipath PD: Bipath sublevelset filtration

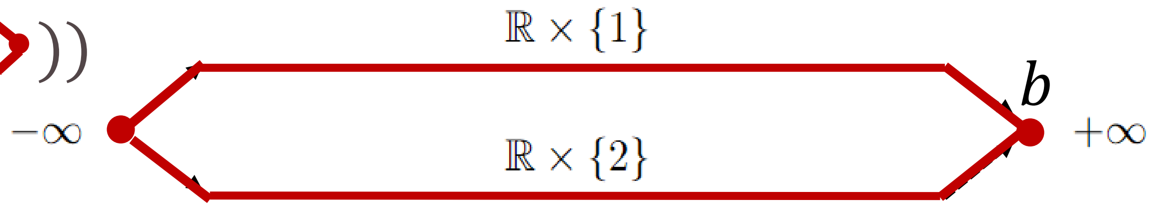
Definition (Bipath sublevelset filtration)

Let f be a bipath function on a top. sp. X. For any b in B, let

$$(f \leq b) := \begin{cases} X & \text{if } b = +\infty \\ f_1^{-1}(\{-\infty\}) & \text{if } b = -\infty \\ f_1^{-1}([-\infty, r] \times \{1\}) & \text{if } b = (r, 1) \\ f_2^{-1}([-\infty, r] \times \{2\}) & \text{if } b = (r, 2) \end{cases}$$

Then, they give a functor $(f \leq \cdot) : B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

$(f \leq +\infty) = X(= f_1^{-1}($  $))$



Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath sublevelset filtration)

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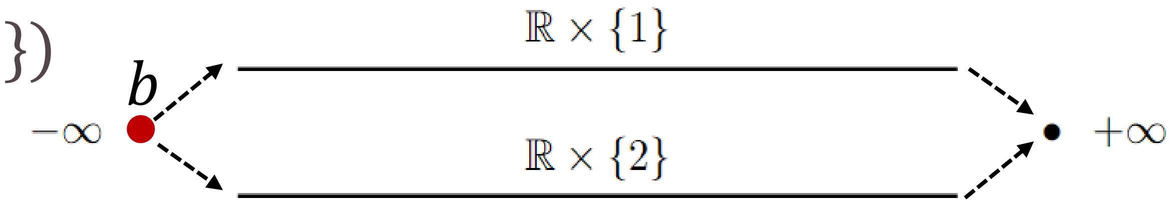
Then, they give a functor $(f \leq \cdot) : B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

$(f \leq -\infty) = f_1^{-1}(\bullet = \{-\infty\})$



Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath sublevelset filtration)

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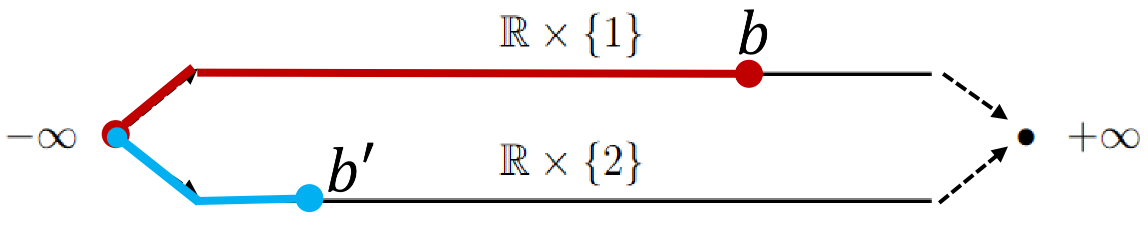
Then, they give a functor $(f \leq \cdot) : B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

$(f \leq b) = f_1^{-1}($  $)$

$(f \leq b') = f_2^{-1}($  $)$



Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath sublevelset filtration)

Let f be a bipath function on a top. sp. X. For any b in B, let

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Then, they give a functor $(f \leq \cdot) : B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

- $f : X \to B$: A bipath function.
- $V_f := H_q(\,\cdot\,; k) \circ (f \leq \cdot)$: Bipath PH of the sublevelset filtration of f.
- $\mathcal{B}(V_f)$:Bipath persistence diagram.

Stability theorem for bipath PD: Theorem

- $f : X \to B$: A bipath function.
- $V_f := H_q(\,\cdot\,; k) \circ (f \leq \cdot)$: Bipath PH of the sublevelset filtration of f.

Condition

For two bipath functions f and g, we set the following condition:

$$f_1^{-1}(\{-\infty\}) \overset{\text{def}}{=} f_2^{-1}(\{-\infty\}) \overset{\blacklozenge}{=} g_1^{-1}(\{-\infty\}) \overset{\text{def}}{=} g_2^{-1}(\{-\infty\}) \qquad (\blacklozenge)$$

Theorem [T, 25](Stability theorem for bipath PD).

Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be bipath functions satisfying ($\blacklozenge$). Then, we have the following inequality:

$$d_{\mathrm{B}}(\mathcal{B}(V_f), \mathcal{B}(V_g)) \leq ||f - g||_\infty.$$

- $||f - g||_\infty := \max\{||f_1 - g_1||_\infty, ||f_2 - g_2||_\infty\}.$

Summary

We proposed a new setting called bipath PH, which is an extension of standard PH.

	Bipath PH
Interval-Decomposability	O
Visualization(Bipath PD)	O
Algorithm(Implementation)	O
Stability theorem for Bipath PD	O
Application	-

Discussion

・Settings where bipath PH is essentially needed.

・Faster algorithms are needed.

・Is an assumption on stability theorem strong?

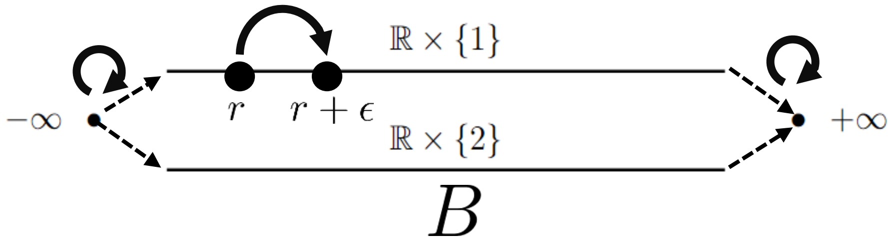
Thank you for listening!

Stability theorem for bipath PD: Bottleneck distance

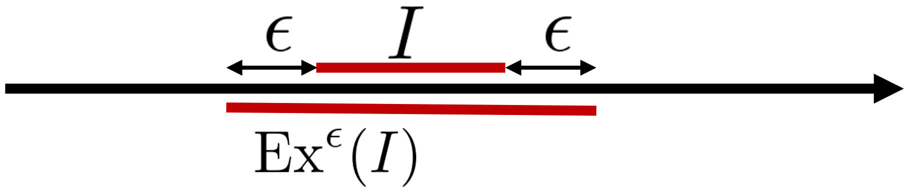
Notation

- $\Lambda := \{\Lambda_\epsilon : B \to B\}_{\epsilon \in \mathbb{R}}$: A family of order-preserving maps satisfying:

  $\Lambda_\epsilon(\pm\infty) = \pm\infty$, and $\Lambda_\epsilon((r,i)) = (r+\epsilon, i)$ for $(r,i) \in \mathbb{R} \times \{i\}$.



- $\mathrm{Ex}^\epsilon(I) := \bigcup_{r \in [-\epsilon, \epsilon]} \Lambda_r(I)$ for an interval I.



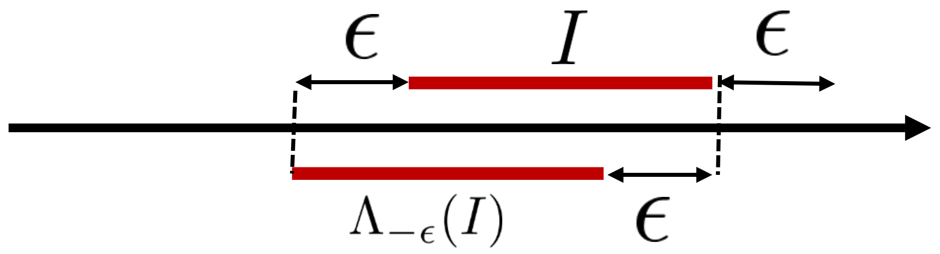
Stability theorem for bipath PD: Bottleneck distance

Proposition

For an interval I, we have $I \cap \Lambda_{-\epsilon}(I) = \emptyset$ if and only if

$I \subseteq \mathbb{R} \times \{1\}$ with $I \cap \{(r-\epsilon, 1) \mid (r, 1) \in I\} = \emptyset$, or
$I \subseteq \mathbb{R} \times \{2\}$ with $I \cap \{(r-\epsilon, 2) \mid (r, 2) \in I\} = \emptyset$.

In this case, we say that I is <u>*ϵ-trivial.*</u>



if $I \subseteq \mathbb{R} \times \{i\}$ ($i = 1$ or $i = 2$)

Stability theorem for bipath PD: Bottleneck distance

Definition (ϵ-matching)

Let X and Y be multisets of intervals. We define ϵ-matching between X and Y by a partial matching $\sigma : X \supseteq X' \xrightarrow{1:1} Y' \subseteq Y$ satisfying:

- Every $I \in X \setminus X'$ is 2ϵ-trivial.
- Every $J \in Y \setminus Y'$ is 2ϵ-trivial.
- If $\sigma(I) = J$, then $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$.

Definition (Bottleneck distance)

The bottleneck distance d_{B} between a multisets of intervals in B is given by

$$d_{\mathrm{B}}(X, Y) := \inf\{\epsilon \geq 0 \mid \exists \epsilon\text{-matching between } X \text{ and } Y\}$$

for multisets of intervals X and Y in B.

Introduction

Theorem [Aoki-Escolar-T, 23]

Let P be a connected finite poset. The following are equivalent.

(a) Every persistence module V over P is interval-decomposable.

(b) The Hasse diagram of P is one of the following forms:

$$A_n(a):\ 1 \longleftrightarrow \cdots \longleftrightarrow n$$

Zigzag posets (type A)

$$B_{n,m}:\ \begin{array}{ccccc} & 1 \rightarrow 2 \overset{\leq}{\rightarrow} \cdots \rightarrow n & \\ \nearrow & & \searrow \\ \hat{0} & & \hat{1} \\ \searrow & \leq & \nearrow \\ & 1' \rightarrow 2' \rightarrow \cdots \rightarrow m' & \end{array}$$

Bipath posets