A Computation of Bipath Persistent Homology and Bipath Persistence Diagrams

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An aim is talking about

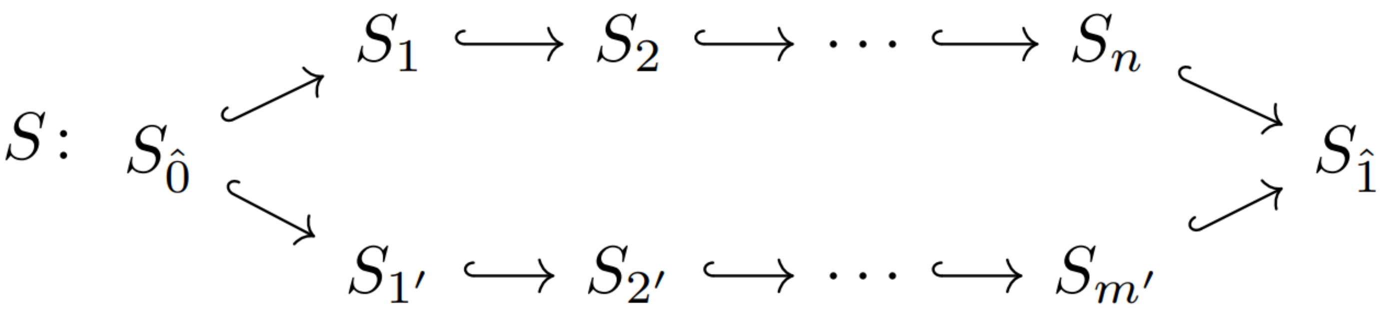
Bipath persistent homology

which is a generalization of standard persistent homology.

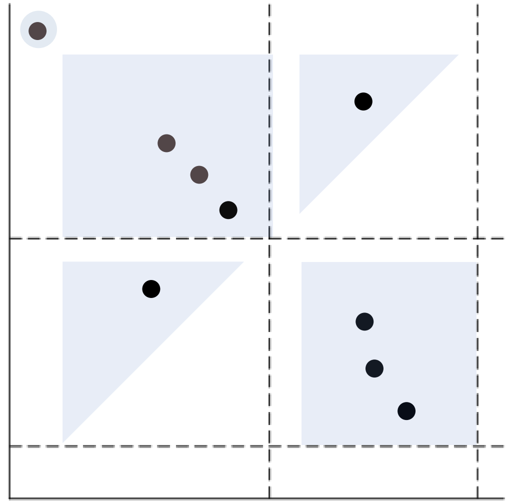
- A Visualization/Computation of Bipath Persistence Diagram

(Joint work with Toshitaka Aoki, Emerson G. Escolar; arXiv:2404.02536)

- Stability Theorem (My currently work)



Bipath filtration

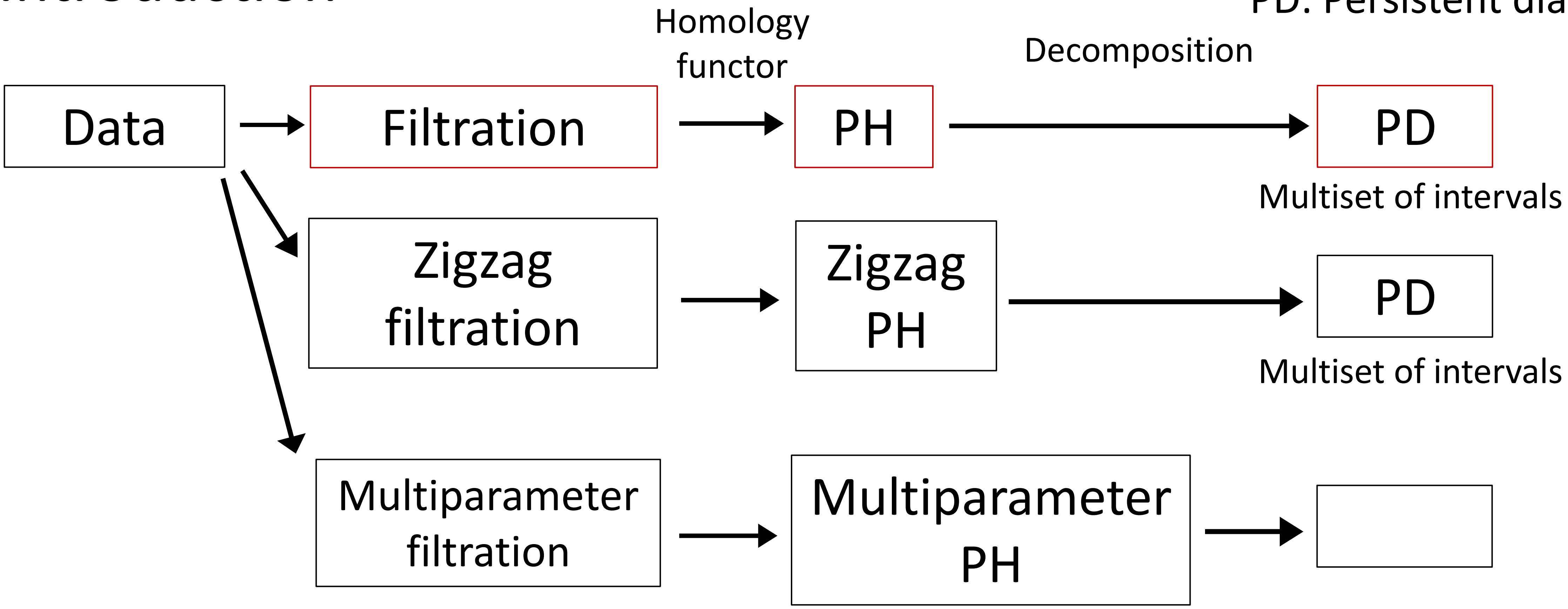


Bipath PD

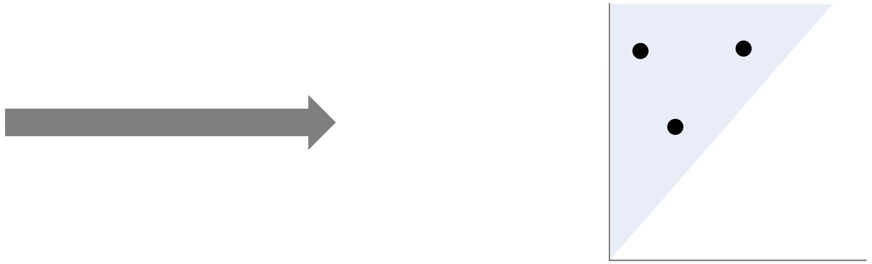
Introduction

PH: Persistent homology

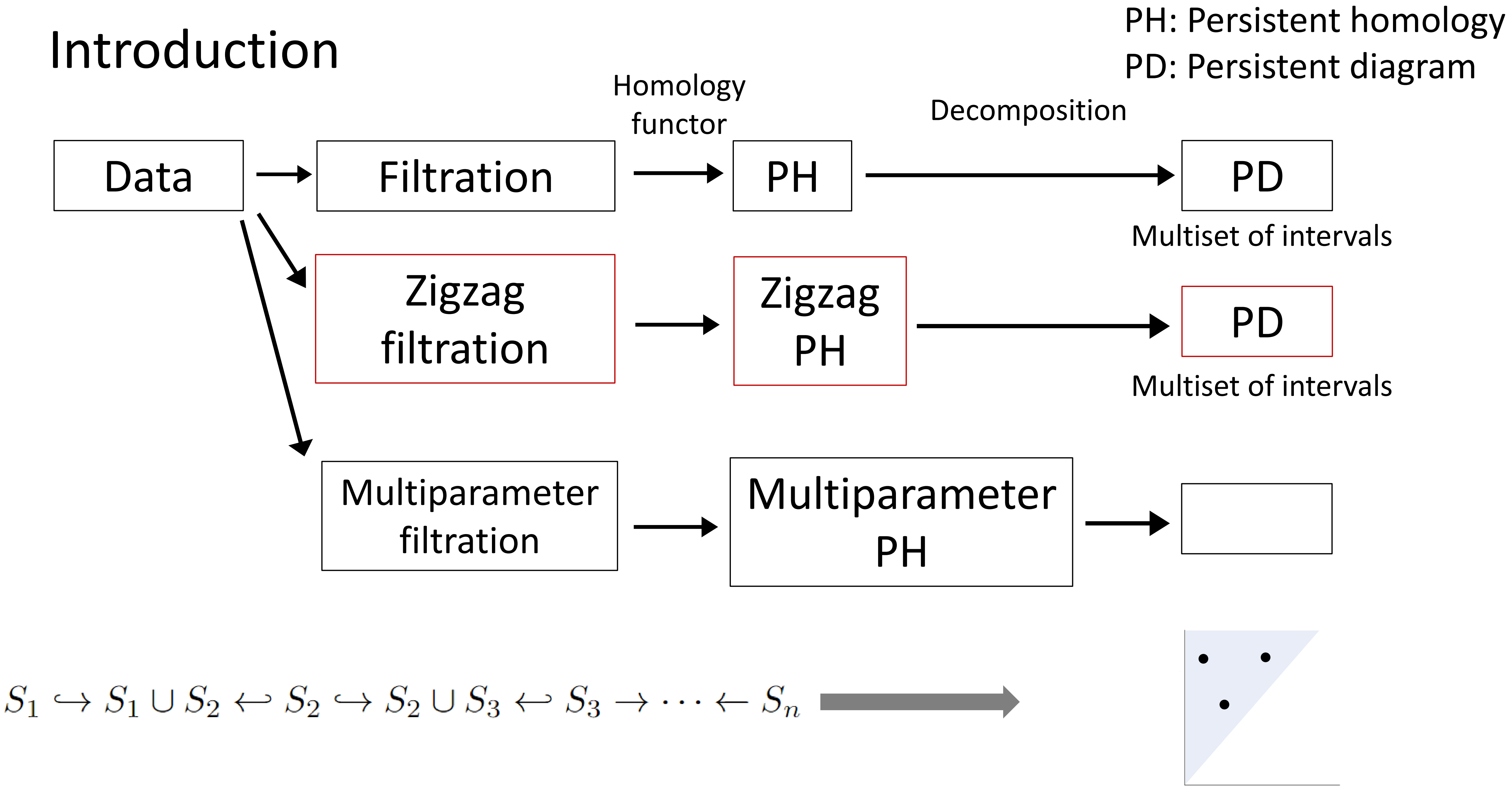
PD: Persistent diagram



$$S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n$$

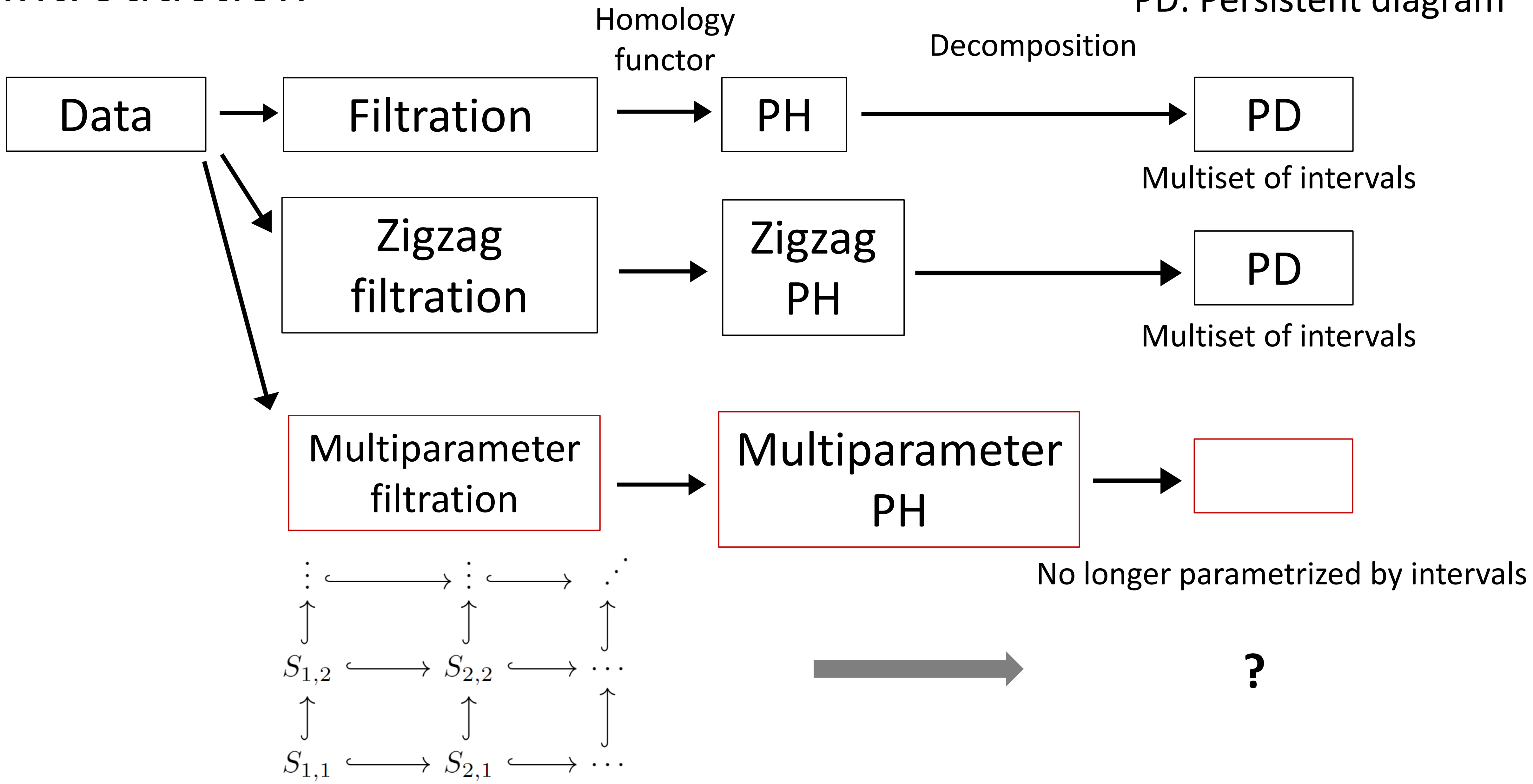


Introduction



Introduction

PH: Persistent homology
PD: Persistent diagram



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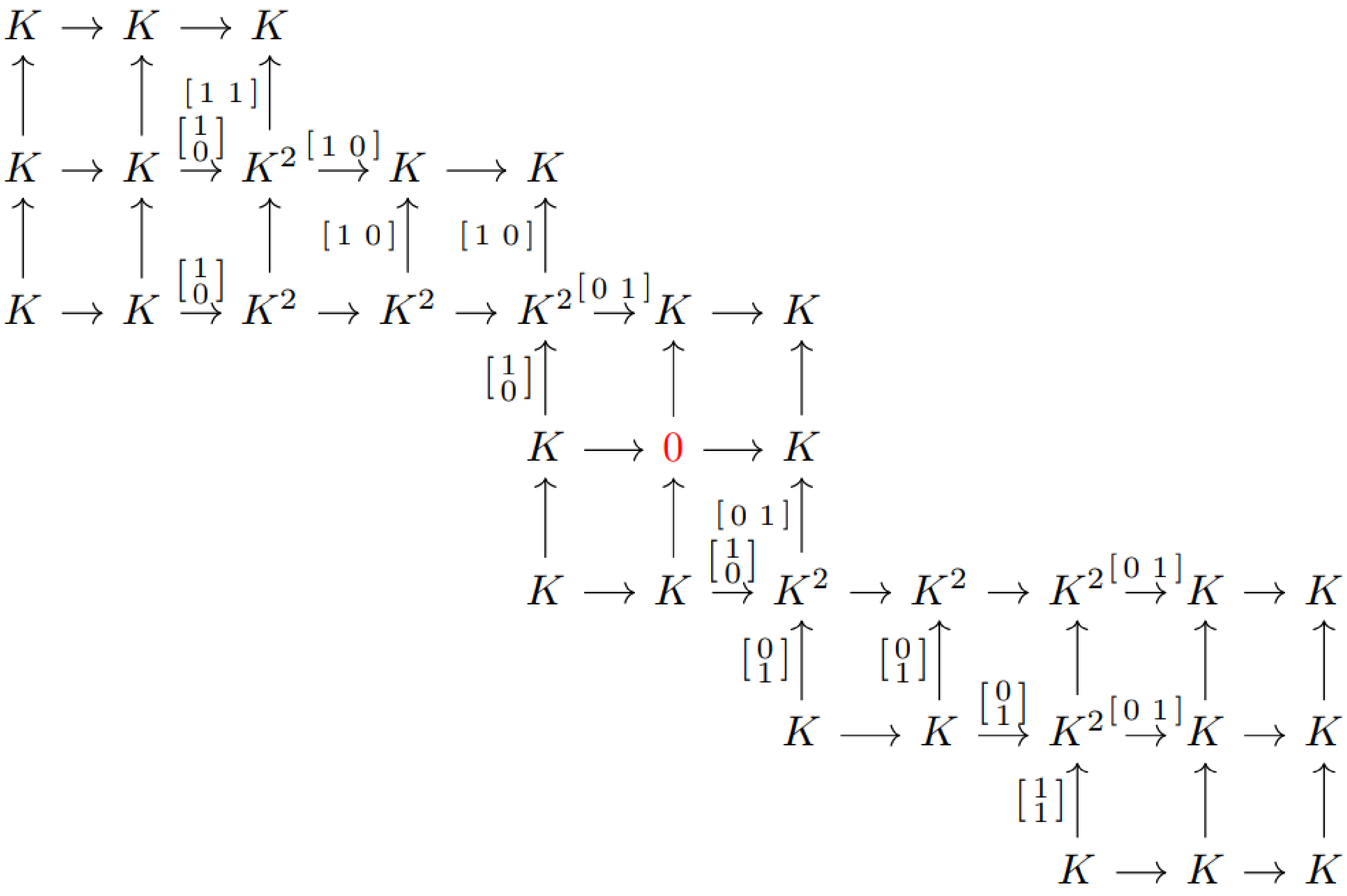


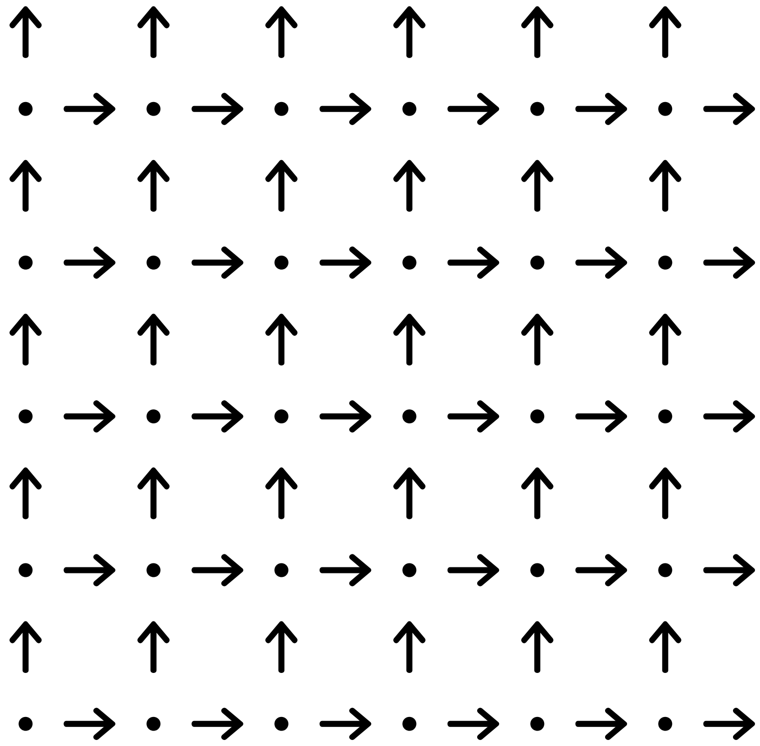
An Indecomposable module:

This is not interval.

M.Buchet, Emerson G. Escolar "Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module" *Journal of Applied and Computational Topology*

Introduction

It is complicated to classify all the indecomposable module.
(wild representation type)



Zbigniew Leszczyński. On the representation type of tensor product algebras. Fundamenta Mathematicae, 144(2):143–161, 1994.
Zbigniew Leszczyński and Andrzej Skowroński. Tame triangular matrix algebras. Colloquium Mathematicum, 86(2):259 303, 2000.
Ulrich Bauer, Magnus B Botnan, Steffen Oppermann, and Johan Steen. Cotorsion torsion triples and the representation theory of filtered hierarchical clustering. Advances in Mathematics, 369:107171, 2020.

Introduction

PH: Persistent homology

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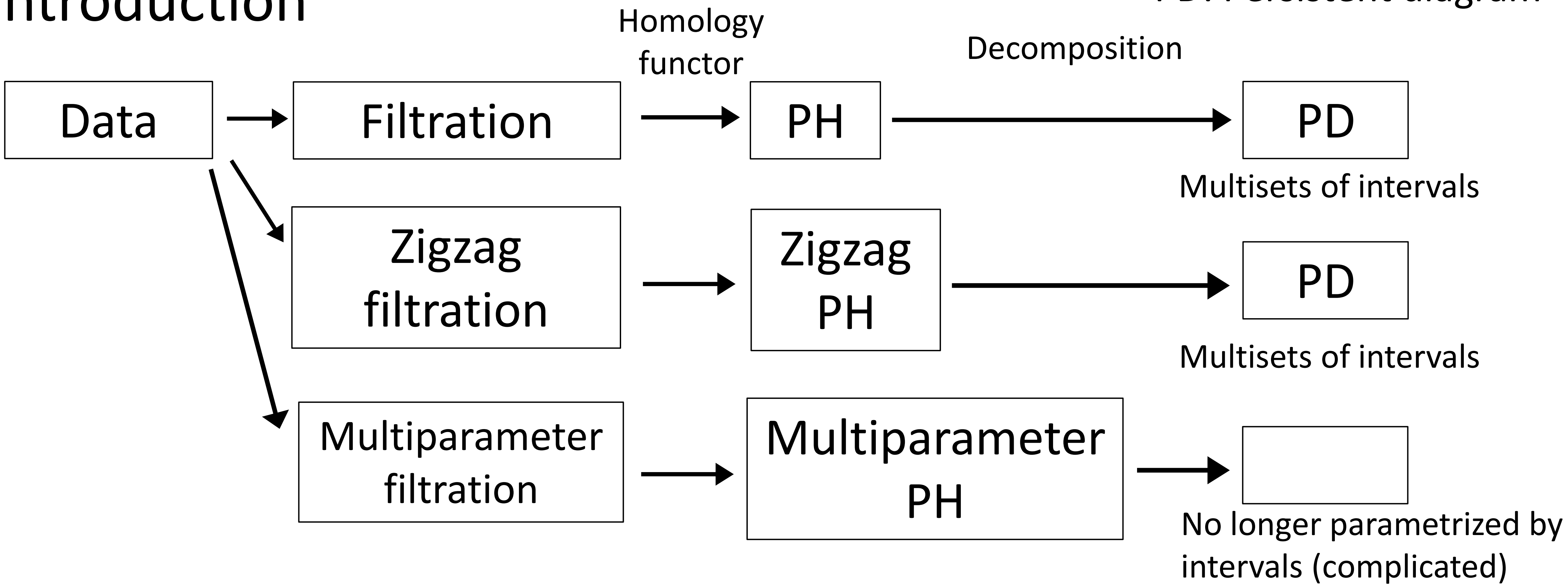
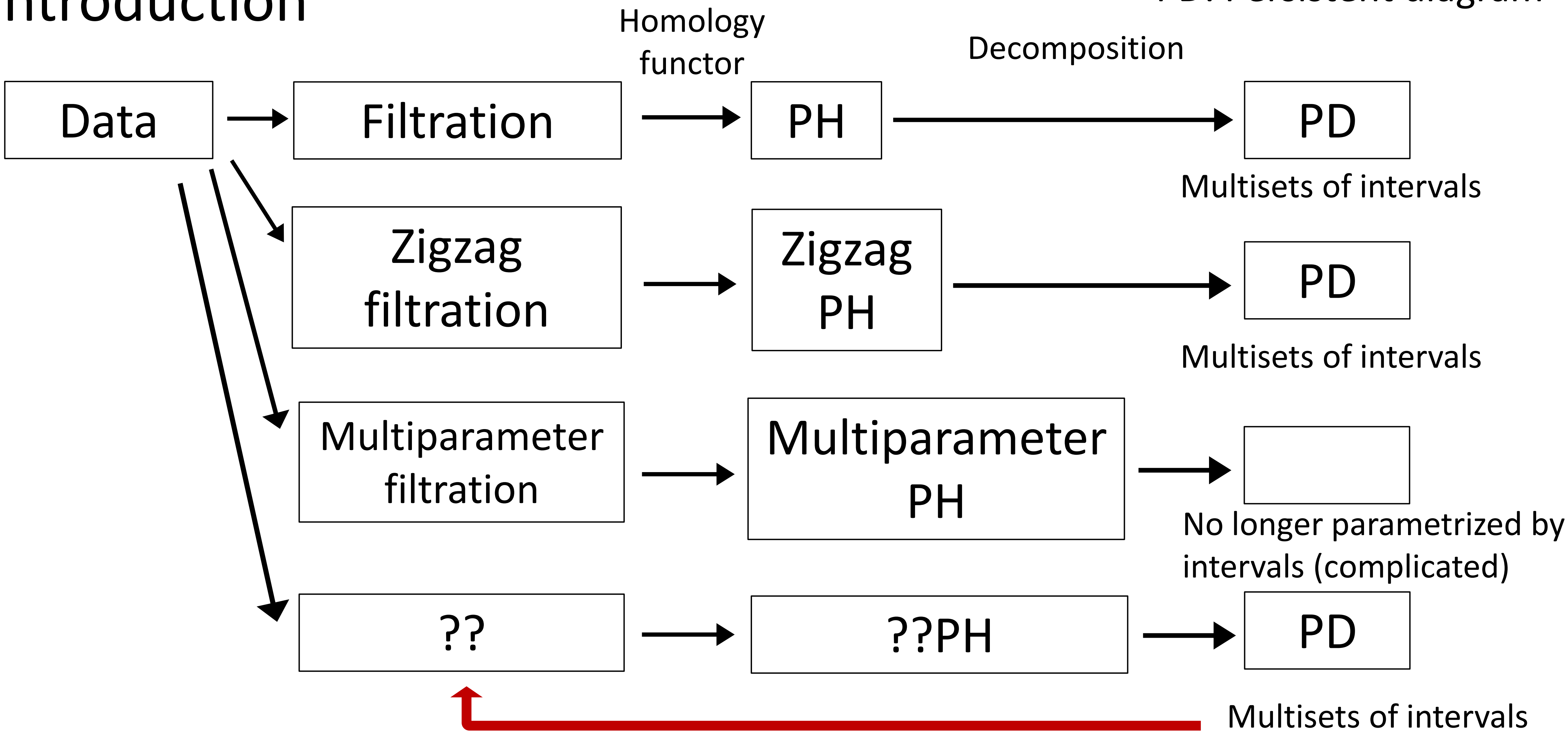


PH: Persistent homology
PD: Persistent diagram

Introduction



Do we have other arrangement of spaces like standard/zigzag filtration?

Introduction

Theorem [Aoki-Escolar-T, 23]

Let P be a connected finite poset. The following are equivalent.

(a) Every persistence module V over P is interval-decomposable.
(b) The Hasse diagram of P is one of the following forms:

$$A_n(a)\colon\ 1 \longleftrightarrow \cdots \longleftrightarrow n$$

Zigzag posets (type A)

$$B_{n,m}\colon\ \begin{array}{ccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & \leq & \searrow \\ \hat{0} & & \hat{1} \\ \searrow & \leq & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

Bipath posets

Introduction

$$
S: \quad \begin{array}{ccccccccc}
 & \nearrow & S_1 & \hookrightarrow & S_2 & \hookrightarrow \cdots \hookrightarrow & S_n & \searrow & \\
S_{\hat{0}} & & & & & & & & S_{\hat{1}} \\
 & \searrow & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow \cdots \hookrightarrow & S_{m'} & \nearrow &
\end{array}
$$

Bipath filtration

||

$$
S: \quad \begin{array}{c}
S_{\hat{0}} \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{\hat{1}} \\
\| \qquad\qquad\qquad\qquad\qquad\qquad \| \\
S_{\hat{0}} \hookrightarrow S_{1'} \hookrightarrow S_{2'} \hookrightarrow \cdots \hookrightarrow S_{m'} \hookrightarrow S_{\hat{1}}
\end{array}
$$

A pair of two filtrations sharing the same spaces at the ends.

Introduction

We can consider a *bipath persistent homology* (bipath PH) of a filtration.

$$
\begin{array}{ccccccccccc}
 & & S_1 & \hookrightarrow & S_2 & \hookrightarrow & \cdots & \hookrightarrow & S_n & & \\
 & \nearrow & & & & & & & & \searrow & \\
S_{\hat{0}} & & & & & & & & & & S_{\hat{1}} \\
 & \searrow & & & & & & & & \nearrow & \\
 & & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow & \cdots & \hookrightarrow & S_{m'} & &
\end{array}
$$

$$\Downarrow$$

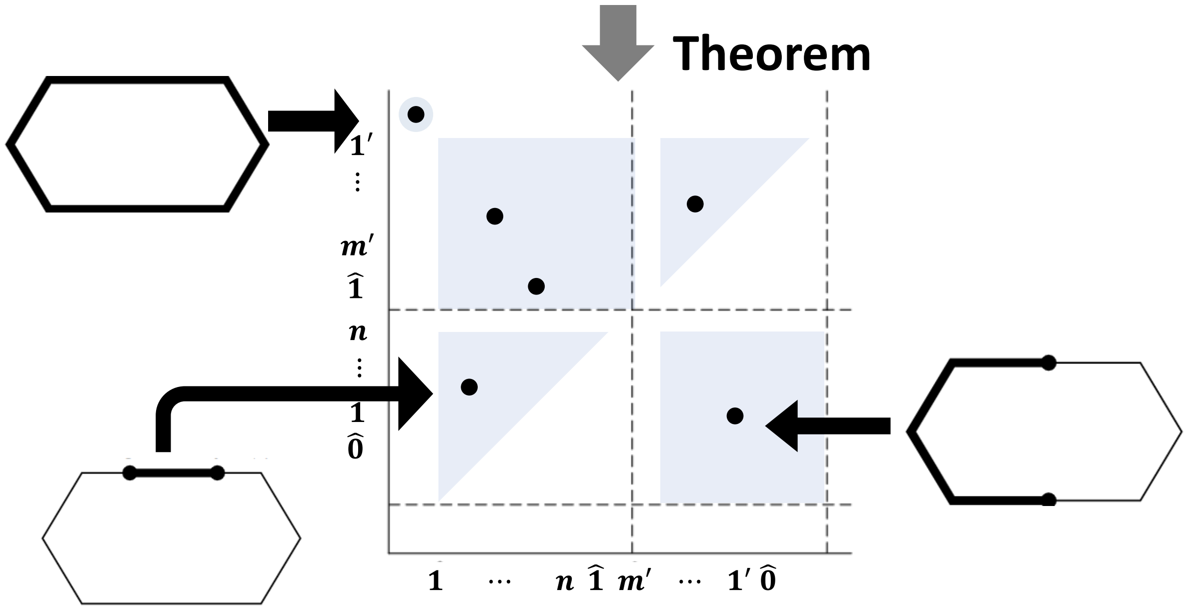
$$
\begin{array}{ccccccccccc}
 & & H_q(S_1;k) & \longrightarrow & H_q(S_2;k) & \longrightarrow & \cdots & \longrightarrow & H_q(S_n;k) & & \\
 & \nearrow & & & & & & & & \searrow & \\
H_q(S_{\hat{0}};k) & & & & & & & & & & H_q(S_{\hat{1}};k) \\
 & \searrow & & & & & & & & \nearrow & \\
 & & H_q(S_{1'};k) & \longrightarrow & H_q(S_{2'};k) & \longrightarrow & \cdots & \longrightarrow & H_q(S_{m'};k) & &
\end{array}
$$

Bipath PH

Introduction

We can get a *Bipath Persistence Diagram* (Bipath PD).

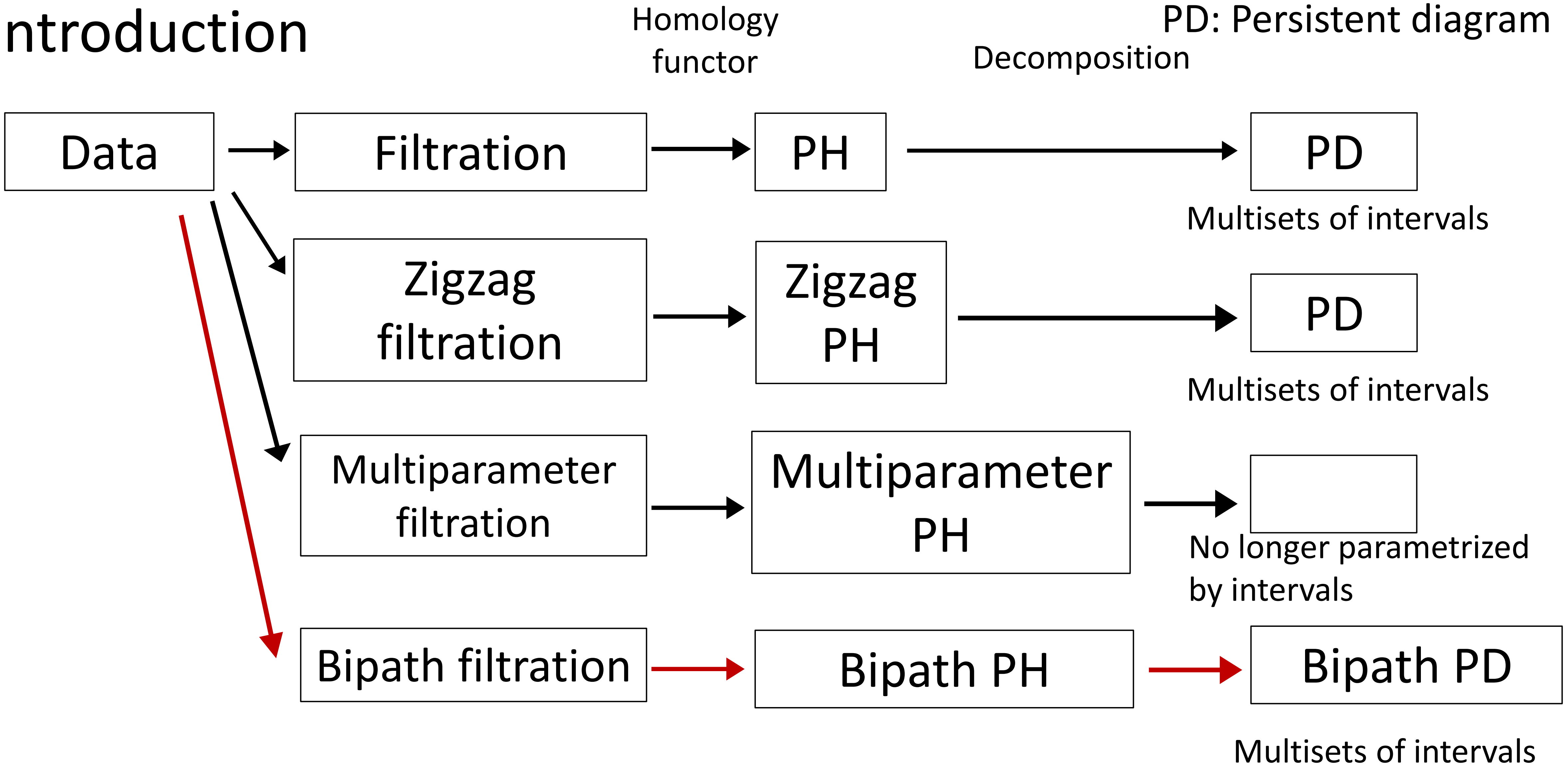
$$
\begin{array}{ccccccccc}
 & \nearrow & H_q(S_1;k) & \longrightarrow & H_q(S_2;k) & \longrightarrow \cdots \longrightarrow & H_q(S_n;k) & \searrow & \\
H_q(S_{\hat{0}};k) & & & & & & & & H_q(S_{\hat{1}};k) \\
 & \searrow & H_q(S_{1'};k) & \longrightarrow & H_q(S_{2'};k) & \longrightarrow \cdots \longrightarrow & H_q(S_{m'};k) & \nearrow &
\end{array}
$$



Bipath PD

PH: Persistent homology
PD: Persistent diagram

Introduction



Introduction

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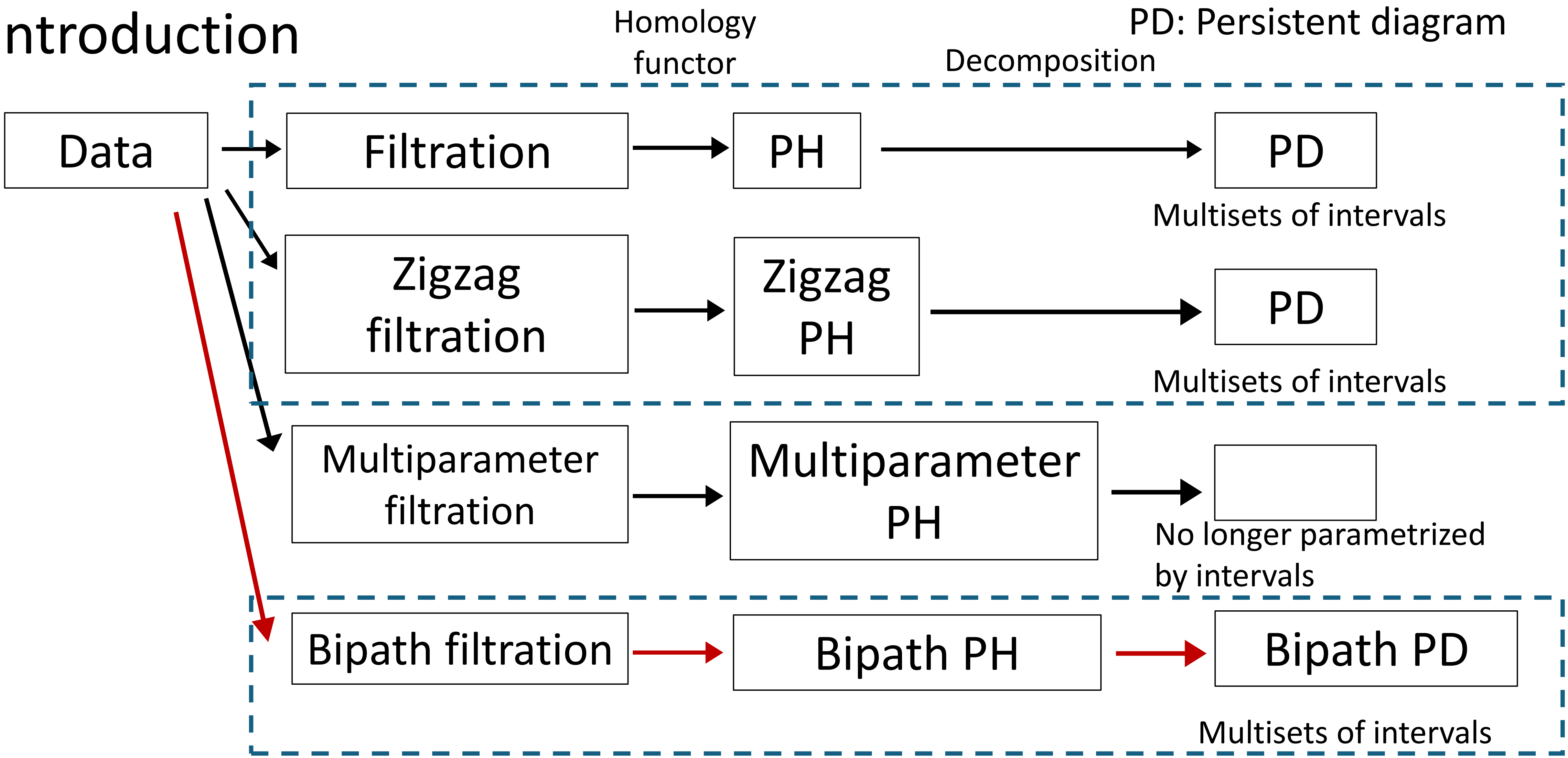
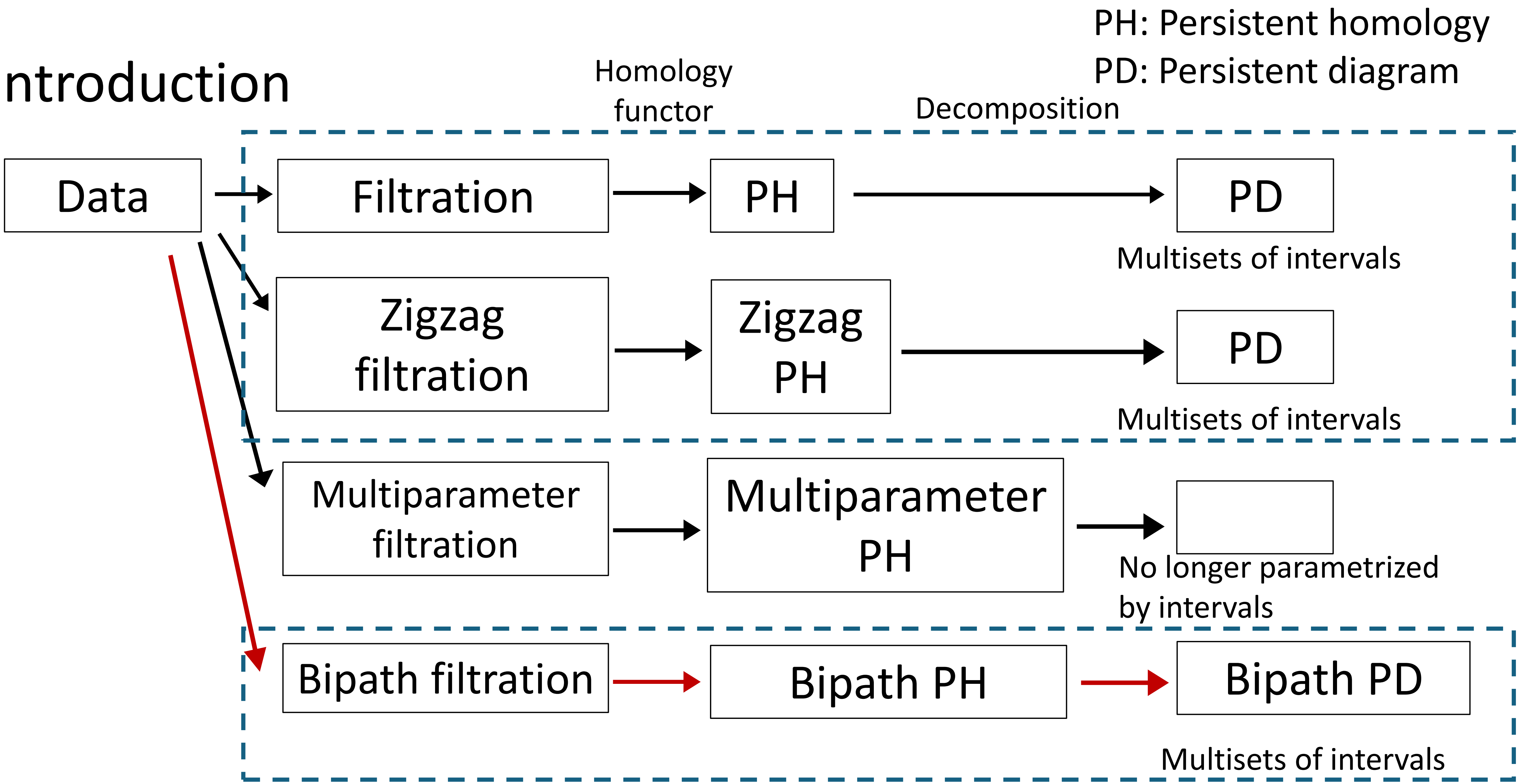
PD: Persistent diagram



Introduction



->What can bipath PH do?

Introduction

Bipath PH can be used to…
(1) study the persistence of topological features across a pair of filtrations connected at their ends, to compare the two filtrations.

$$
\begin{array}{ccccccccccccc}
 & S_{\hat{0}} & \hookrightarrow & S_1 & \hookrightarrow & S_2 & \hookrightarrow & \cdots & \hookrightarrow & S_n & \hookrightarrow & S_{\hat{1}} \\
S: & \| & & & & & & & & & & \| \\
 & S_{\hat{0}} & \hookrightarrow & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow & \cdots & \hookrightarrow & S_{m'} & \hookrightarrow & S_{\hat{1}}
\end{array}
$$

Introduction

Bipath PH can be used to…
(2) obtain an invariant for multiparameter PH.

$$
\begin{array}{ccccccc}
S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
\uparrow & & & & & & \uparrow \\
S_{1,3} & & & & & & S_{4,3} \\
\uparrow & & & & & & \uparrow \\
S_{1,2} & & & & & & S_{4,2} \\
\uparrow & & & & & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}
\quad \subseteq \quad
\begin{array}{ccccccc}
S_{1,4} & \to & S_{2,4} & \to & S_{3,4} & \to & S_{4,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,3} & \to & S_{2,3} & \to & S_{3,3} & \to & S_{4,3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,2} & \to & S_{2,2} & \to & S_{3,2} & \to & S_{4,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S_{1,1} & \to & S_{2,1} & \to & S_{3,1} & \to & S_{4,1}
\end{array}
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Bipath filtration.

Bifiltration.

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\uparrow & & & & \uparrow & & \\
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\end{array}
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Bipath filtration. Bifiltration.

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 & & \uparrow & & & & \uparrow \\
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\uparrow & & & & \uparrow & & \\
S_{1,2} & & S_{2,2} & \rightarrow & S_{3,2} & & \\
\uparrow & & \uparrow & & & & \\
S_{1,1} & \rightarrow & S_{2,1} & & & &
\end{array}
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\uparrow & & \uparrow & & \uparrow & & \uparrow \\
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S_{1,3} & \rightarrow & S_{2,3} & & & & S_{4,3} \\
\uparrow & & & & & & \uparrow \\
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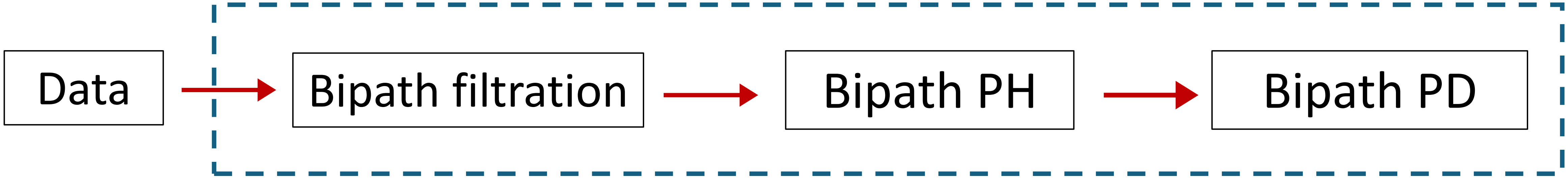
Bipath filtration. Bifiltration.

Introduction



	Bipath PH
Interval-Decomposability	○
Visualization(Bipath PD)	○
Algorithm(Implementation)	○
Stability theorem for Bipath PD	○
Application	-

Implementation

We currently have not yet worked on applications, but we are looking for ideas!

Bipath Persistence

Bipath Persistence

Definition Bipath poset

Let m and n be non-negative numbers. A *bipath poset* $B_{n,m}$ is a poset consisting of two totally ordered sets

$$1 \leq 2 \leq \cdots \leq n, \text{ and } 1' \leq 2' \leq \cdots \leq m'$$

with the global minimum and the global maximum

$$\hat{0} \text{ and } \hat{1}.$$

The Hasse diagram:

$$B_{n,m}: \quad \begin{array}{ccccccccc} & \nearrow & 1 & \rightarrow & 2 & \overset{\leq}{\rightarrow} \cdots \rightarrow & n & \searrow & \\ \hat{0} & & & & & & & & \hat{1} \\ & \searrow & 1' & \rightarrow & 2' & \overset{\leq}{\rightarrow} \cdots \rightarrow & m' & \nearrow & \end{array}$$

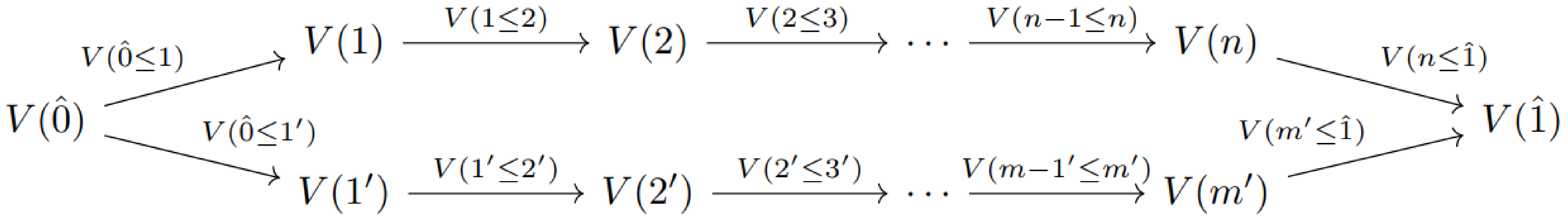
Bipath Persistence

Definition Bipath persistence module

A *bipath persistence module* is a functor from $B_{n,m}$ (regarding as a category) to a category of finite dimensional k-vector spaces.

k: field

*A bipath persistence module V is displayed by



with a commutative relation

$$V(n \leq \hat{1}) \cdots V(1 \leq 2) V(\hat{0} \leq 1) = V(m' \leq \hat{1}) \cdots V(1' \leq 2') V(\hat{0} \leq 1').$$

Bipath Persistence

Definition (Bipath filtration)

A *bipath filtration* S is a functor from $B_{n,m}$ to a category of topological spaces with $S(a \leq b)\colon S(a) \hookrightarrow S(b)$ for any $a \leq b$ in $B_{n,m}$.

*A bipath filtration S is displayed by:

$$
S: \quad \begin{array}{ccccccccc}
 & & S_1 & \hookrightarrow & S_2 & \hookrightarrow \cdots \hookrightarrow & S_n & & \\
 & \nearrow & & & & & & \searrow & \\
S_{\hat{0}} & & & & & & & & S_{\hat{1}} \\
 & \searrow & & & & & & \nearrow & \\
 & & S_{1'} & \hookrightarrow & S_{2'} & \hookrightarrow \cdots \hookrightarrow & S_{m'} & &
\end{array}
$$

Definition (Bipath persistent homology)

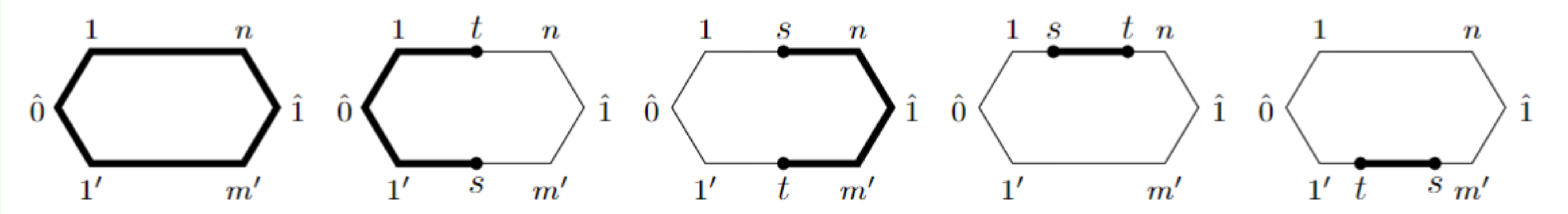
For a bipath filtration S, we obtain a bipath persistence module $H_q(S;k)$ called *bipath persistent homology* of S.

*We consider the case that bipath PH is pointwise finite dimensional.

Bipath Persistence

Intervals in $B := B_{n,m}$ are one of the following forms:

$\mathcal{B}$ (All)	$\mathcal{L}$ (Left)	$\mathcal{R}$ (Right)	$\mathcal{U}$ (Up)	$\mathcal{D}$ (Down)



$\hat{0} \in I$, $\hat{1} \in I$	$\hat{0} \in I$, $\hat{1} \notin I$	$\hat{0} \notin I$, $\hat{1} \in I$	$\hat{0} \notin I$, $\hat{1} \notin I$	$\hat{0} \notin I$, $\hat{1} \notin I$

・Each interval in B (except for B) is written by the pair $\langle s, t\rangle$ ($s, t \in B$) by taking end points of the interval.

→ We obtain an interval module k_I for each interval $I (= \langle s, t\rangle$ or $B)$.

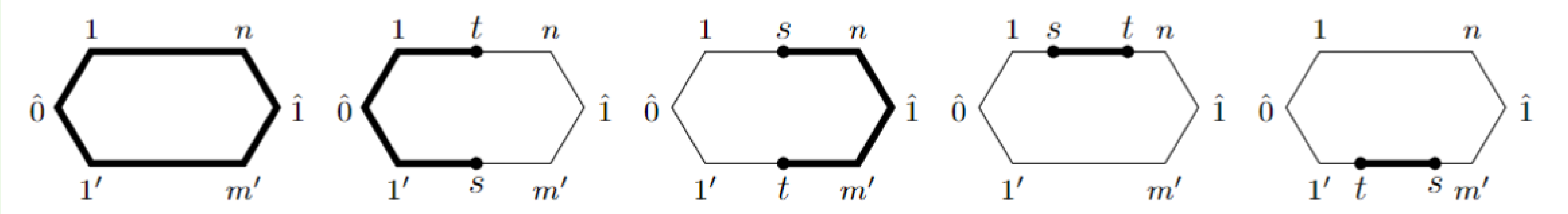
→ $\forall V$: bipath persistence module. $V \cong k_B^{m(B)} \oplus \left(\bigoplus_{\langle s,t\rangle} k_{\langle s,t\rangle}^{m(\langle s,t\rangle)}\right)$

($\langle s, t\rangle$ runs over all intervals in $\mathcal{L} \sqcup \mathcal{R} \sqcup \mathcal{U} \sqcup \mathcal{D}$.)

Bipath Persistence

Intervals in $B := B_{n,m}$ are one of the following forms:

$\mathcal{B}$ (All)	$\mathcal{L}$ (Left)	$\mathcal{R}$ (Right)	$\mathcal{U}$ (Up)	$\mathcal{D}$ (Down)



$\hat{0} \in I$, $\hat{1} \in I$	$\hat{0} \in I$, $\hat{1} \notin I$	$\hat{0} \notin I$, $\hat{1} \in I$	$\hat{0} \notin I$, $\hat{1} \notin I$	$\hat{0} \notin I$, $\hat{1} \notin I$

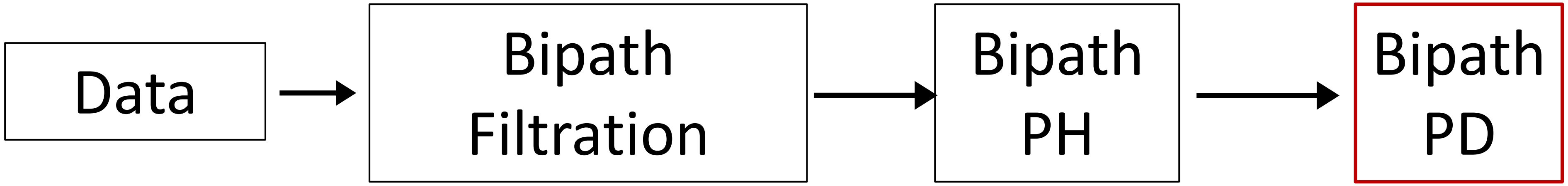
・Each interval in B (except for B) is written by the pair $\langle s, t\rangle$ ($s, t \in B$) by taking end points of the interval.

→ We obtain an interval module k_I for each interval $I(= \langle s, t\rangle$ or B).

→ $\forall V$: bipath persistence module. $V \cong k_B^{m(B)} \oplus (\bigoplus_{\langle s,t\rangle} k_{\langle s,t\rangle}^{m(\langle s,t\rangle)})$

($\langle s, t\rangle$ runs over all intervals in $\mathcal{L} \sqcup \mathcal{R} \sqcup \mathcal{U} \sqcup \mathcal{D}$.) → Bipath PD

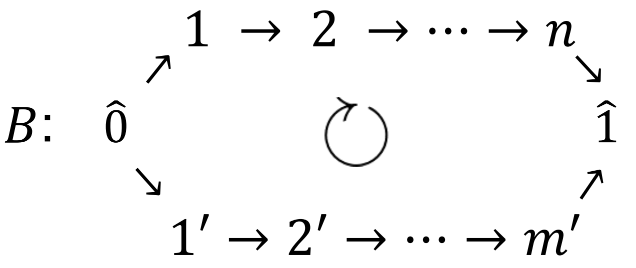
Bipath Persistence Diagram
(Bipath PD)

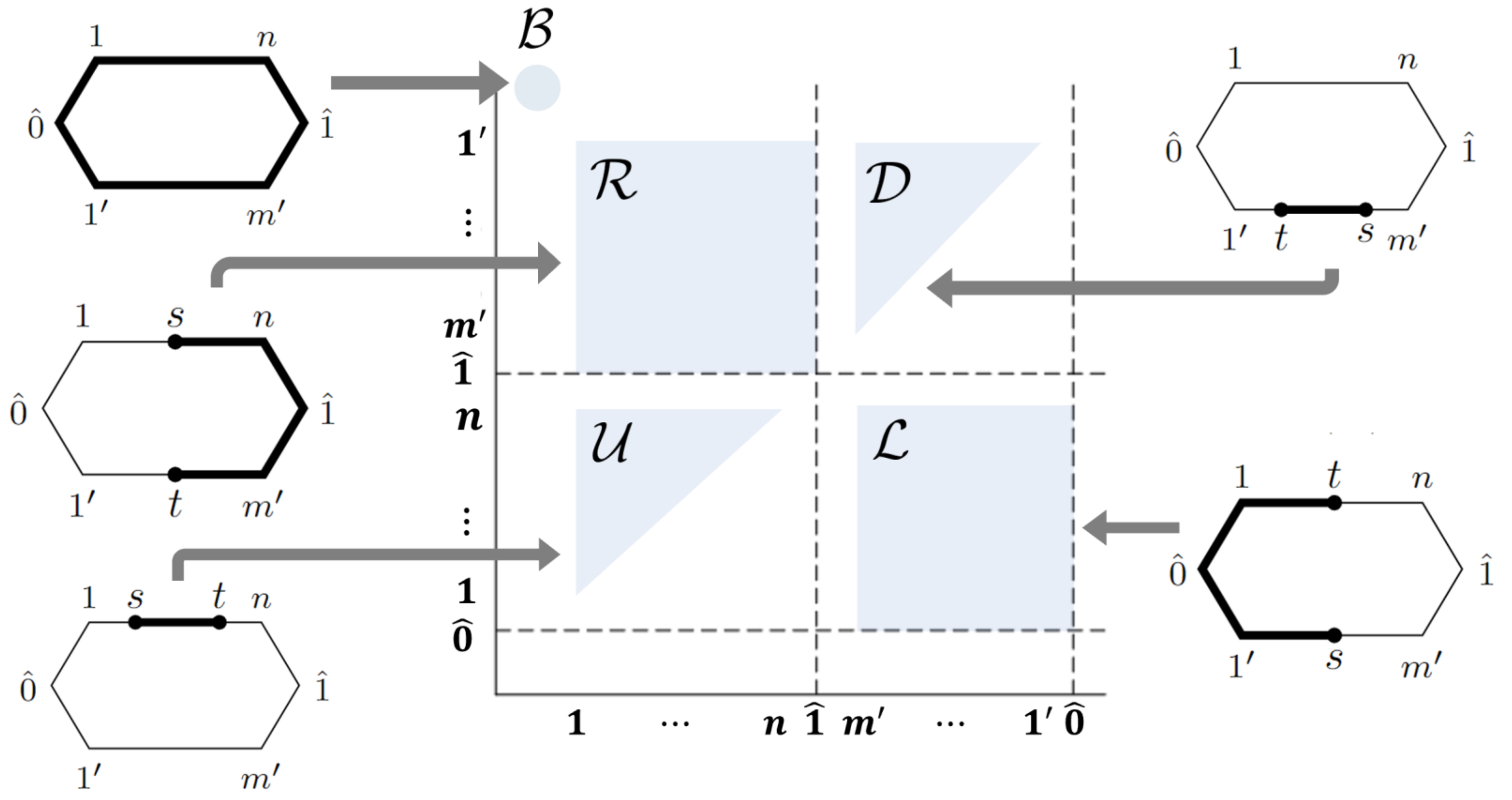


How to visualize.

Bipath PD

(1)Put elements of B (in clockwise) on the vertical and horizontal axes.

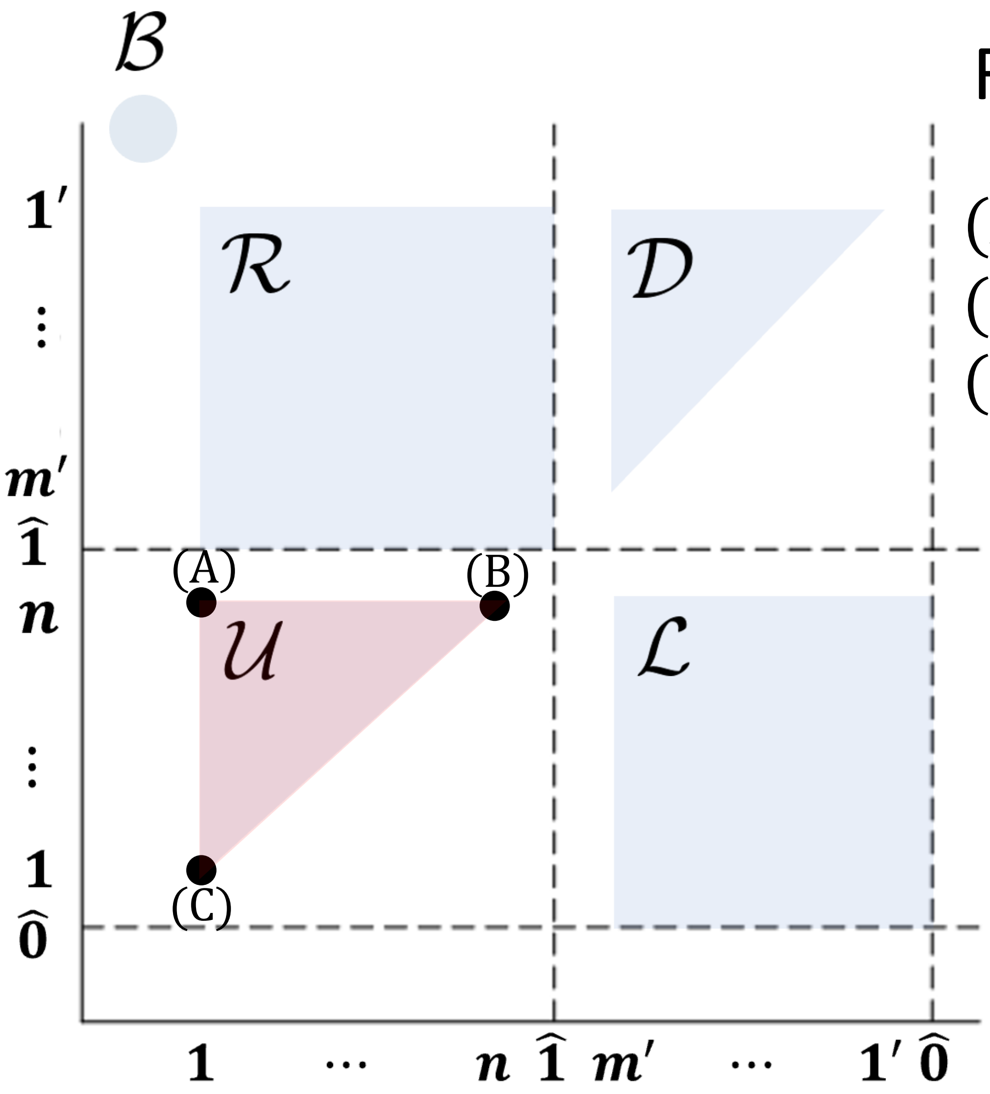




(2)Plot a point on the upper left region "$\mathcal{B}$" for the interval B.

(3)Plot a point (s, t) for the interval $\langle s, t \rangle$.

Bipath PD: Region $\mathcal{U}$

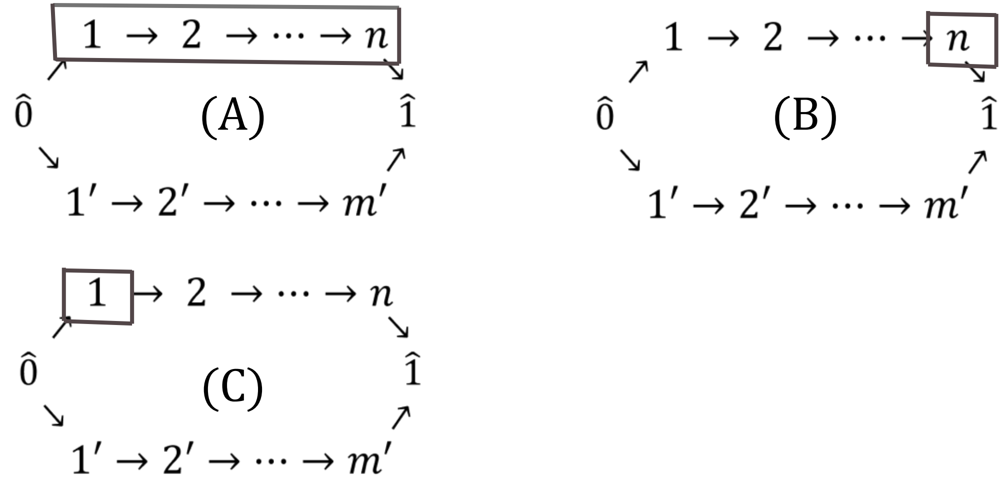


Region $\mathcal{U}$ is for intervals in $\mathcal{U}(B)$.

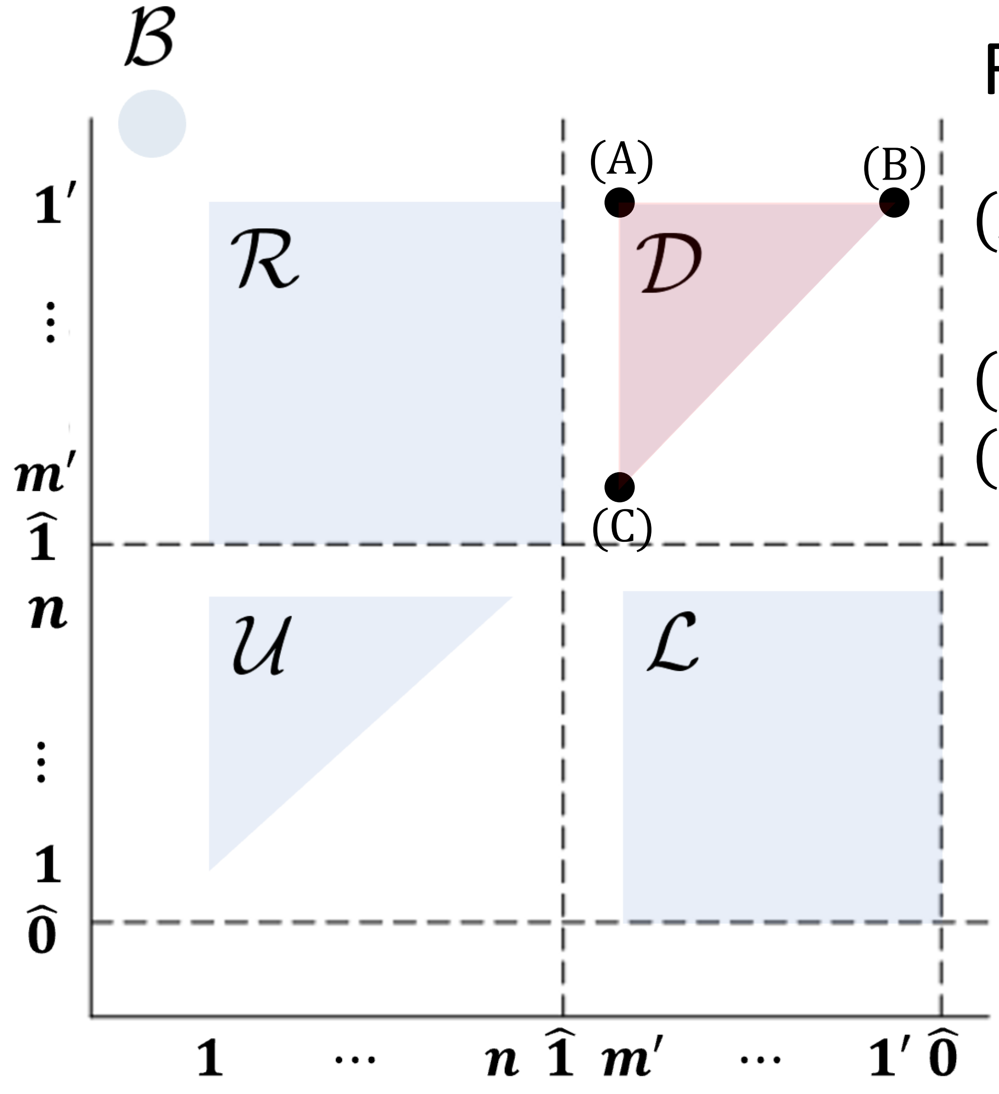
(A) ⋯ $\langle 1, n \rangle = \{1, \dots, n\}$. The longest interval in $\mathcal{U}(B)$.

(B) ⋯ $\langle n, n \rangle = \{n\}$.

(C) ⋯ $\langle 1,1 \rangle = \{1\}$.



Bipath PD: Region $\mathcal{D}$

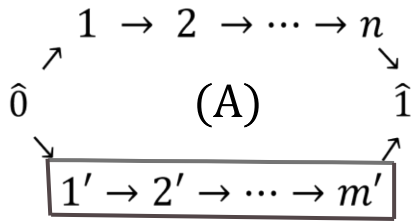


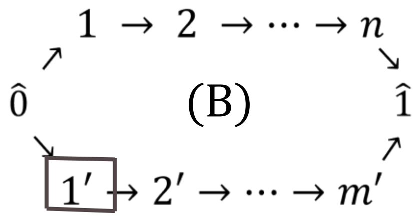
Region $\mathcal{D}$ is for intervals in $\mathcal{D}(B)$.

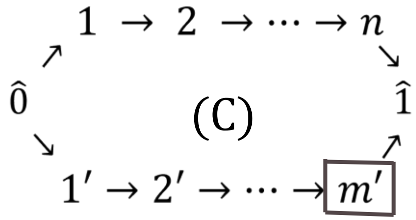
(A)$\cdots \langle m', 1' \rangle = \{m', \ldots, 1'\}$. The longest interval in $\mathcal{D}(B)$.

(B) $\cdots \langle 1', 1' \rangle = \{1'\}$.

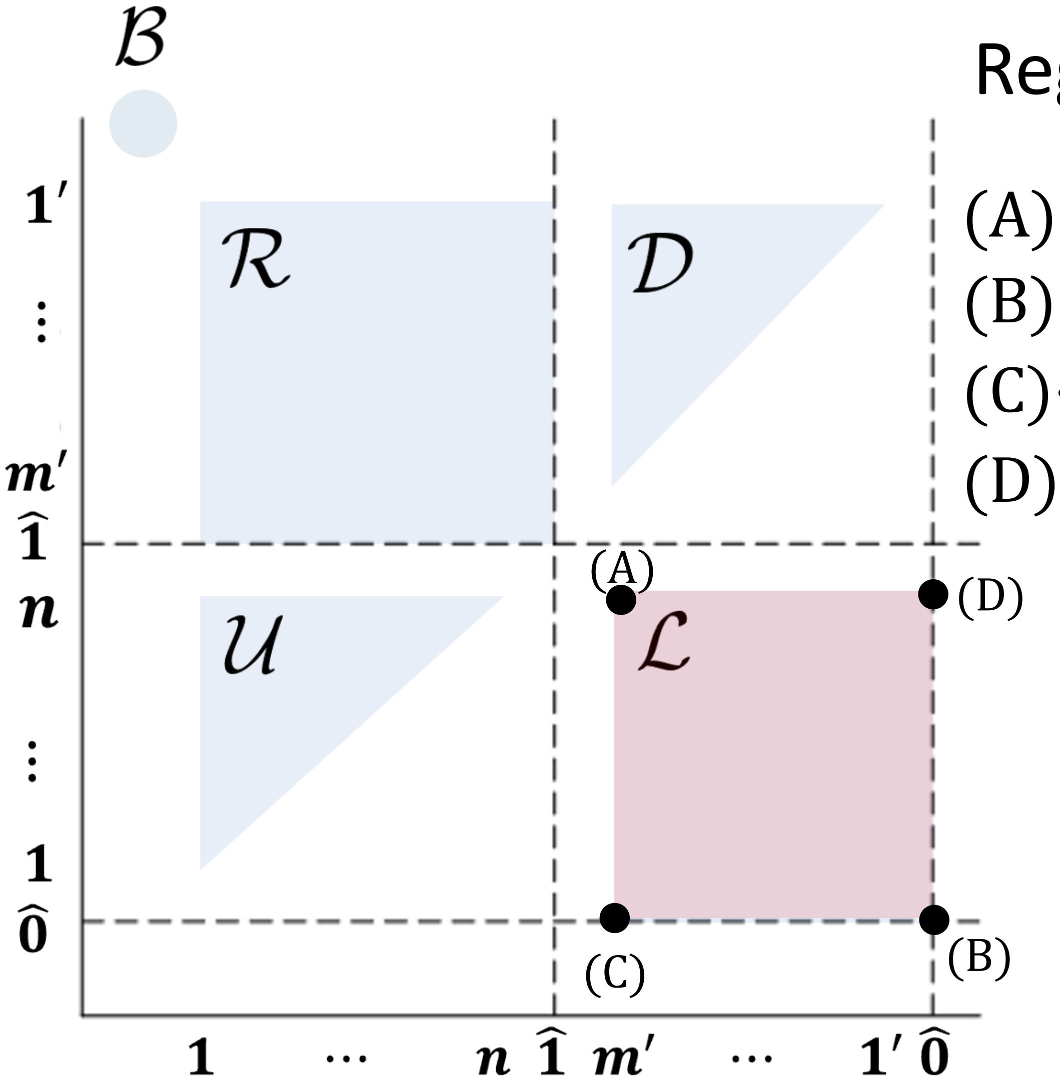
(C) $\cdots \langle m', m' \rangle = \{m'\}$.







Bipath PD: Region $\mathcal{L}$



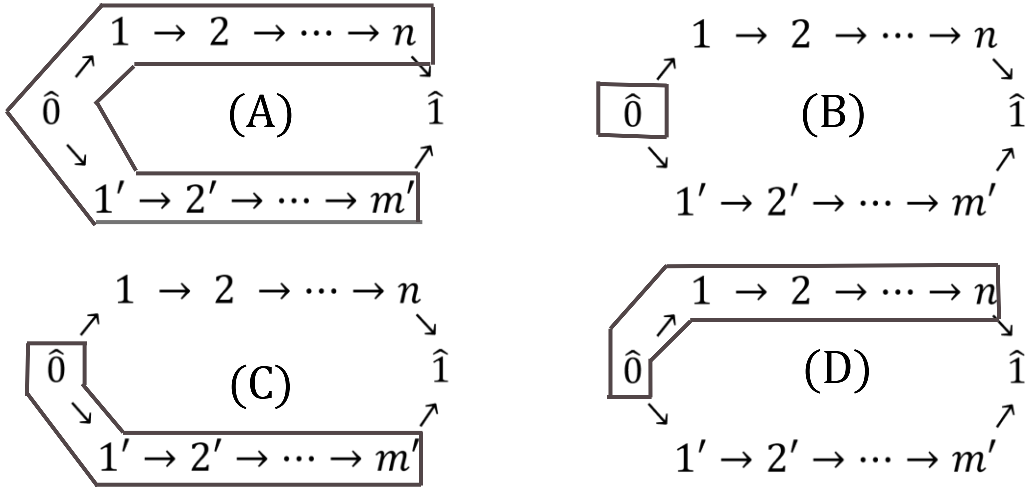
Region $\mathcal{L}$ is for intervals in $\mathcal{L}(B)$.

(A)$\cdots \langle m', n \rangle = B \setminus \{\hat{1}\}$. The longest interval in $\mathcal{L}(B)$.

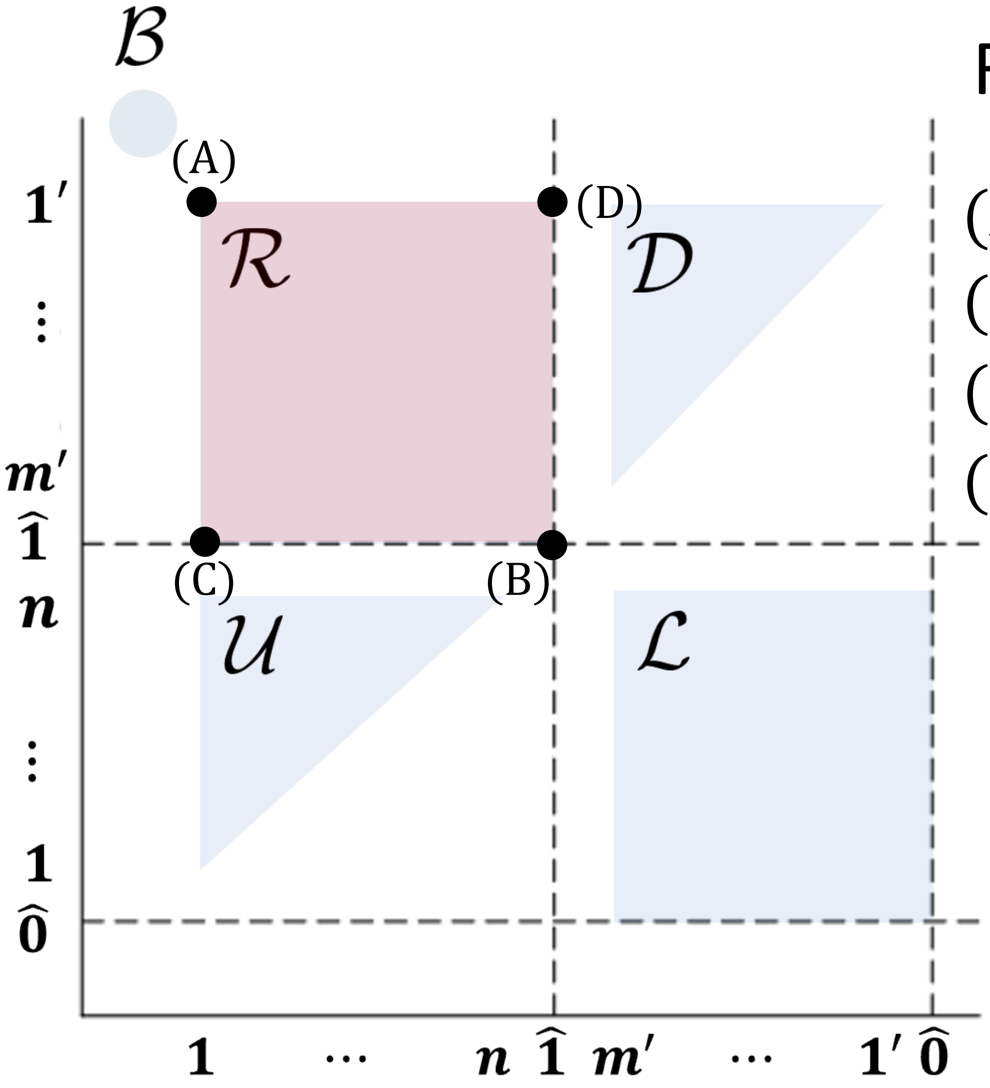
(B)$\cdots \langle \hat{0}, \hat{0} \rangle = \{\hat{0}\}$. The shortest interval in $\mathcal{L}(B)$

(C)$\cdots \langle m', \hat{0} \rangle = \{m', (m-1)', \ldots, \hat{0}\}$.

(D)$\cdots \langle \hat{0}, n \rangle = \{\hat{0}, 1, \ldots, n\}$.



Bipath PD: Region $\mathcal{R}$



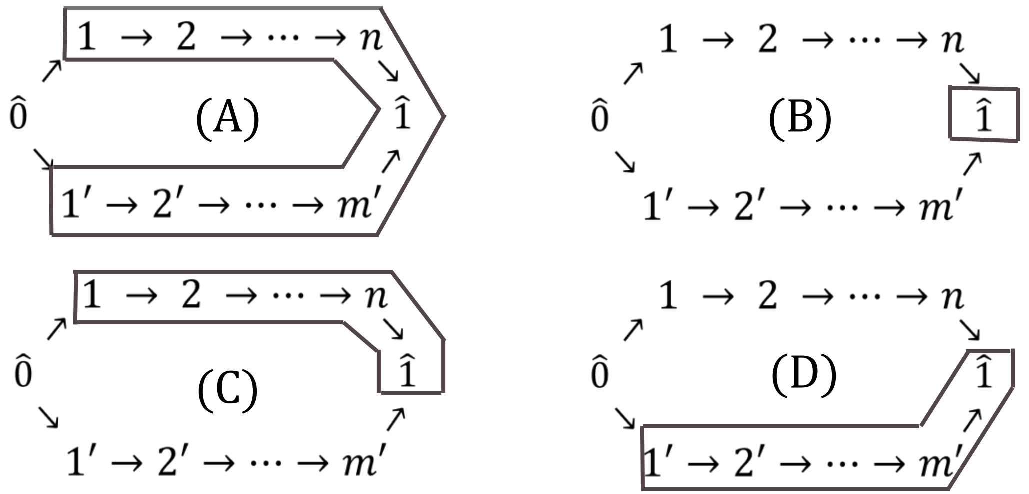
Region $\mathcal{R}$ is for intervals in $\mathcal{R}(B)$.

(A)$\cdots \langle 1,1' \rangle = B \setminus \{\hat{0}\}$. The longest interval in $\mathcal{R}(B)$.

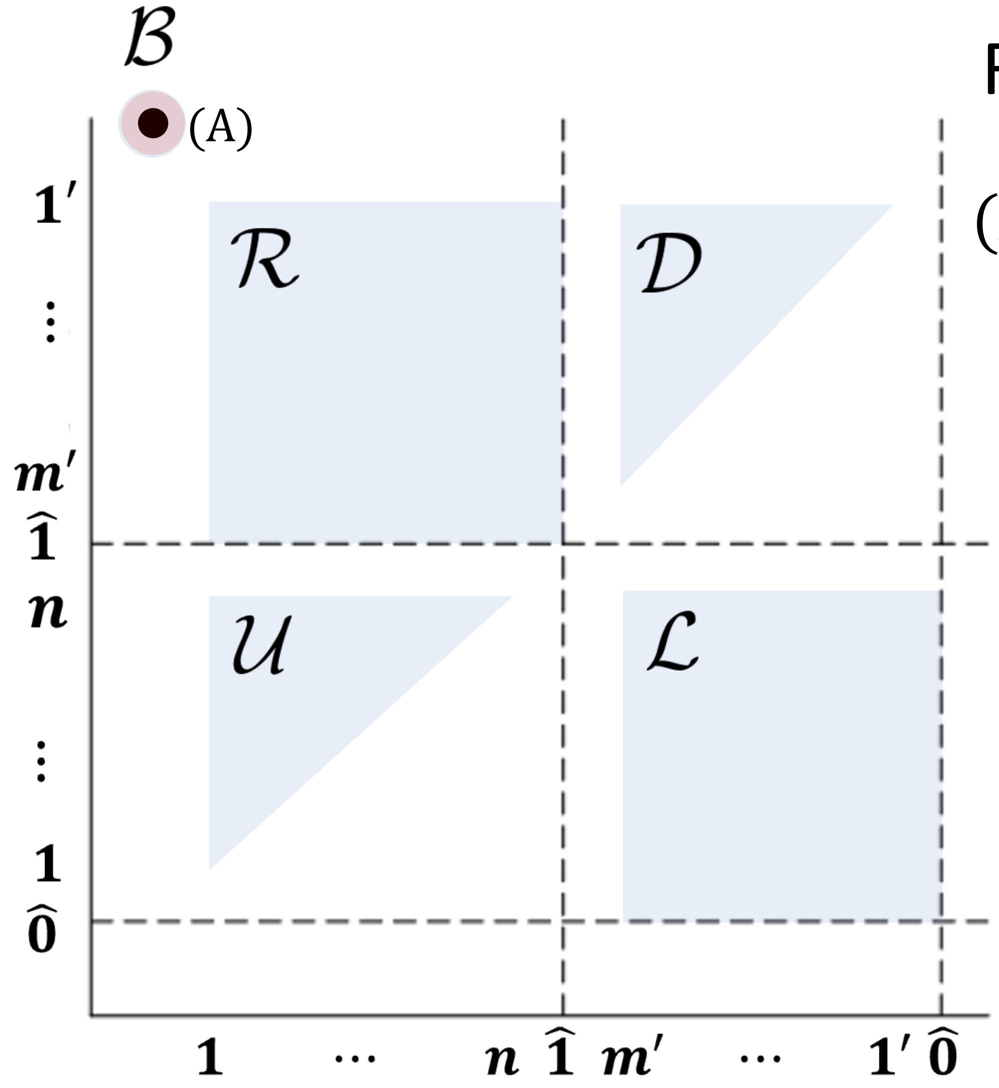
(B)$\cdots \langle \hat{1},\hat{1} \rangle = \{\hat{1}\}$. The shortest interval in $\mathcal{R}(B)$.

(C)$\cdots \langle 1,\hat{1} \rangle = \{m',(m-1)',\ldots,\hat{0}\}$.

(D)$\cdots \langle \hat{1},1' \rangle = \{\hat{0},1,\ldots,n\}$.

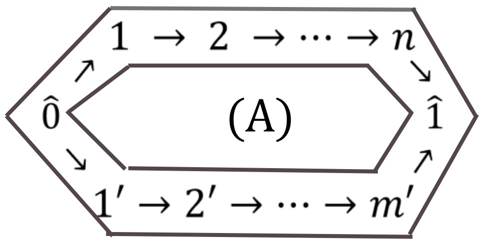


Bipath PD: Region $\mathcal{B}$



Region $\mathcal{B}$ is for intervals in $\mathcal{B}(B)(=\{B\})$.

(A)$\cdots B$.

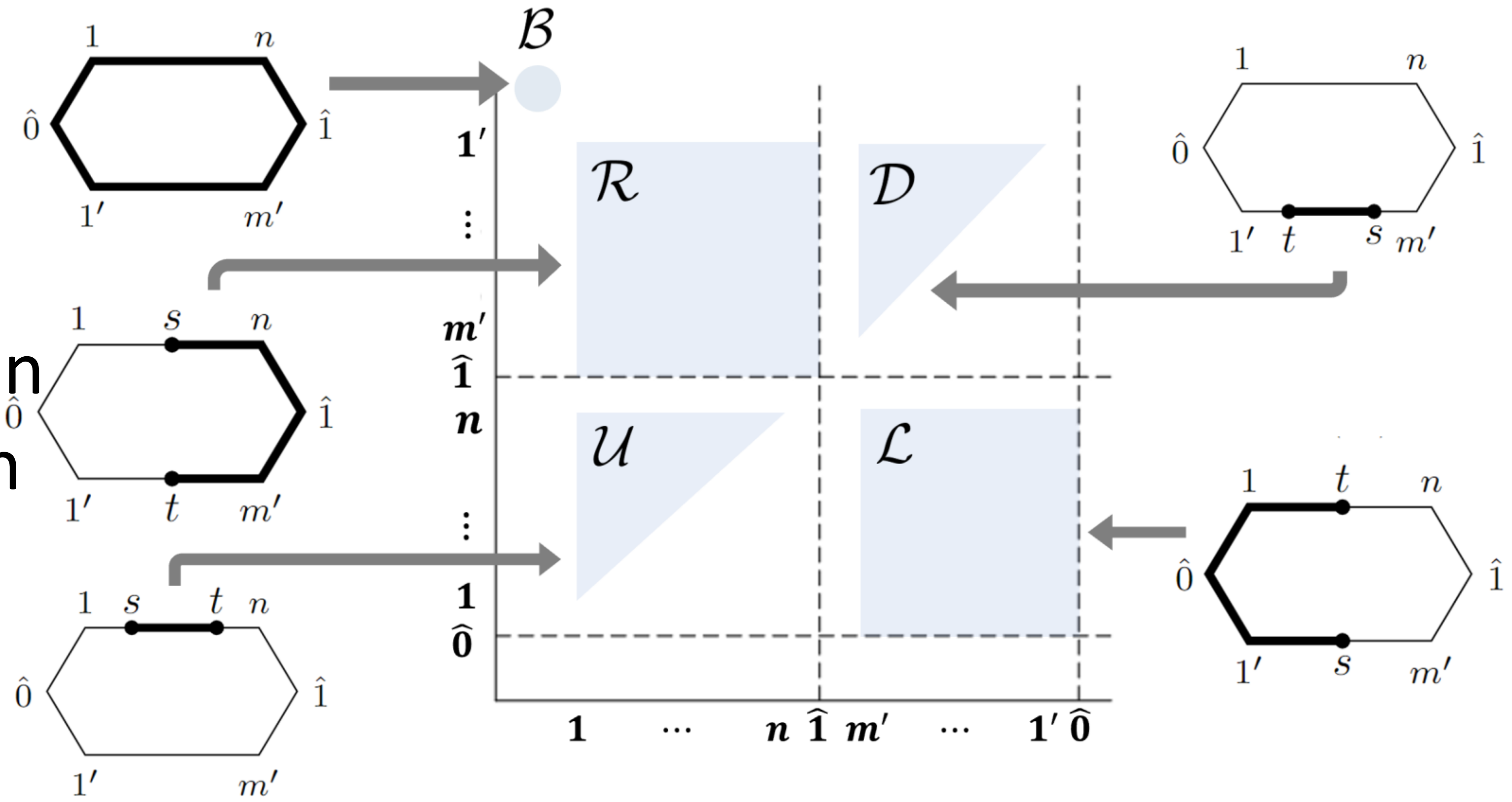


Bipath PD

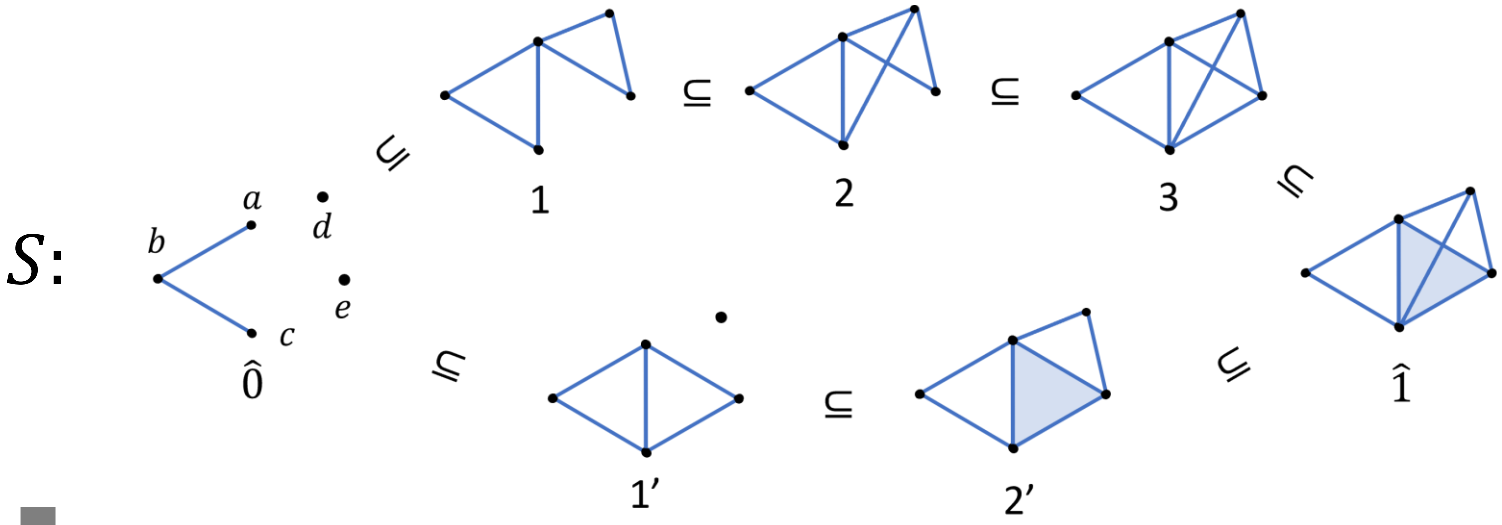
・For each region in the bipath PD, points plotted on left and/or above correspond to intervals with longer length.

・The region $\mathcal{U}$ can be regarded as standard PD.

・For a point (s, t) in region $\mathcal{D}$, s and t represent death and birth respectively. (Opposite to standard notation.)

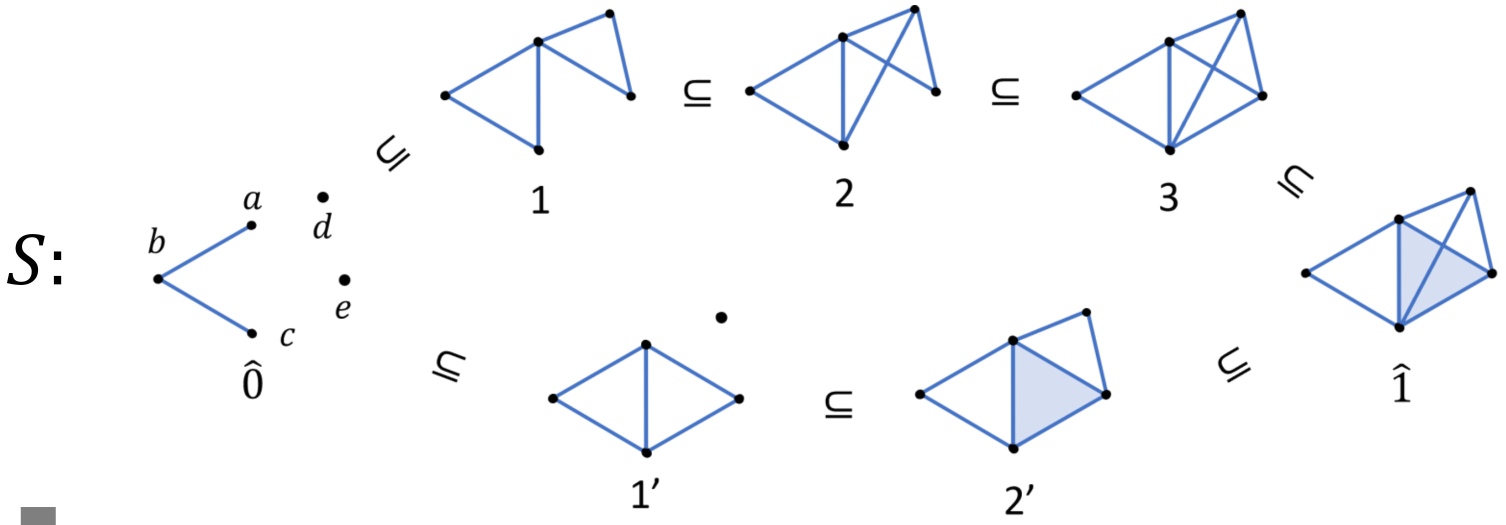


Bipath PD



- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.
- Compare upper and lower filtration.

Bipath PD

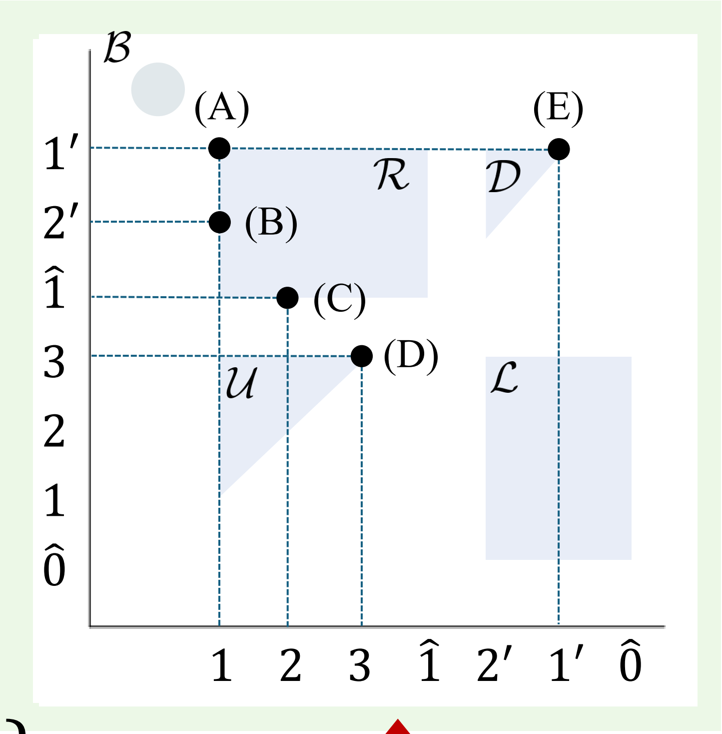


- Compute bipath PD of $H_1(S; k = \mathbb{F}_2)$.
- Compare upper and lower filtration.

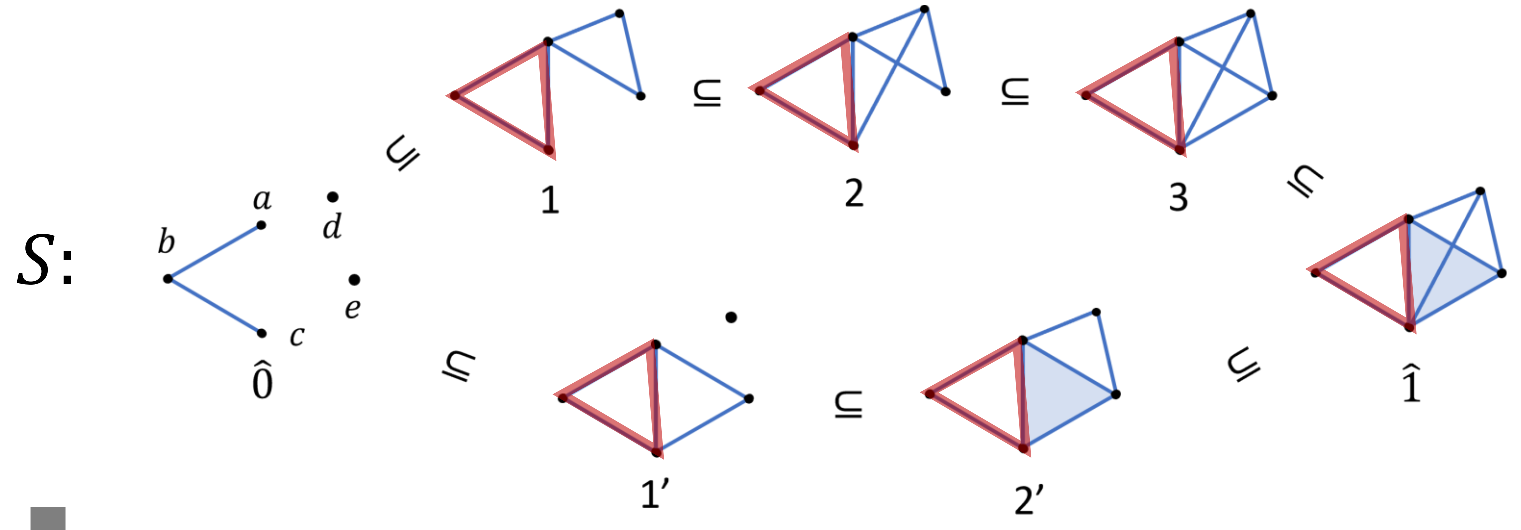
$$\mathcal{B}(H_1(S;k)) = \{\{1,2,3,\hat{1},2',1'\},\{1,2,3,\hat{1},2'\},\{2,3,\hat{1}\},\{3\},\ \{1'\}\}$$

$$= \{\langle 1,1'\rangle, \langle 1,2'\rangle, \langle 2,\hat{1}\rangle, \langle 3,3\rangle, \langle 1',1'\rangle\}.$$

(A) (B) (C) (D) (E)



Bipath PD



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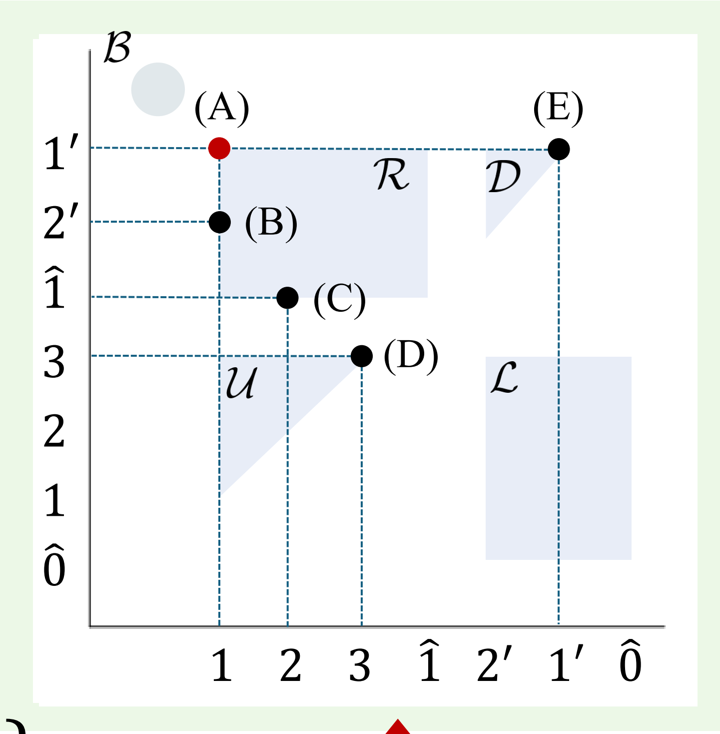
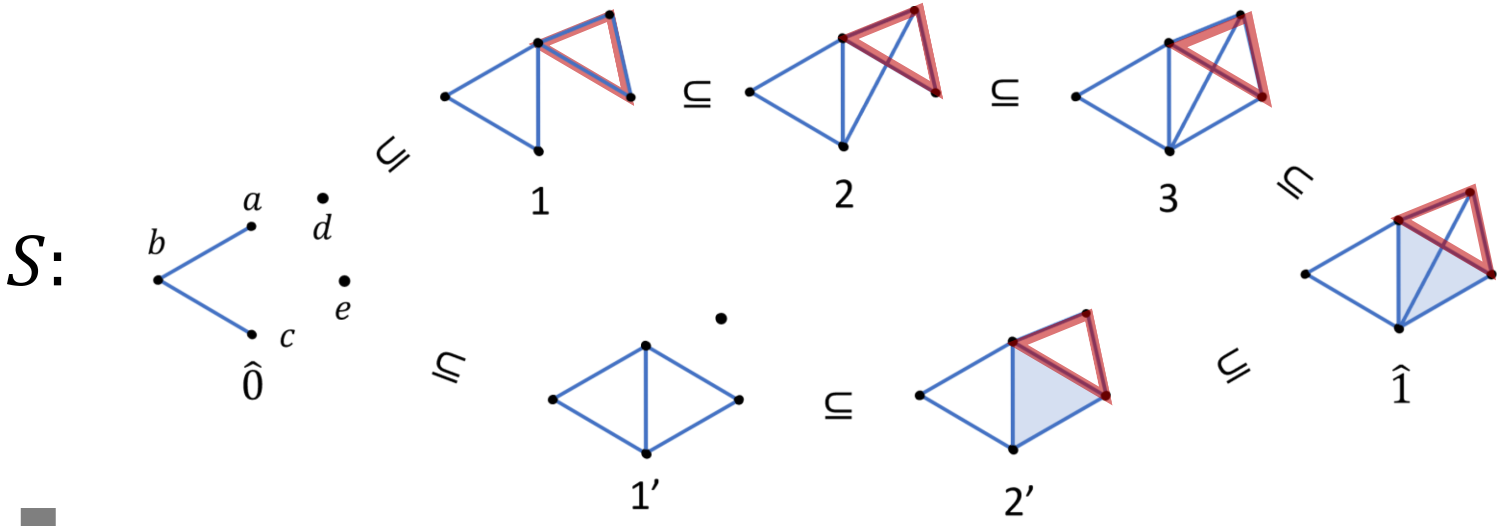
$$= \{\langle 1,1' \rangle, \langle 1,2' \rangle, \langle 2,\hat{1} \rangle, \langle 3,3 \rangle, \langle 1',1' \rangle\}.$$

(A) (B) (C) (D) (E)



Bipath PD



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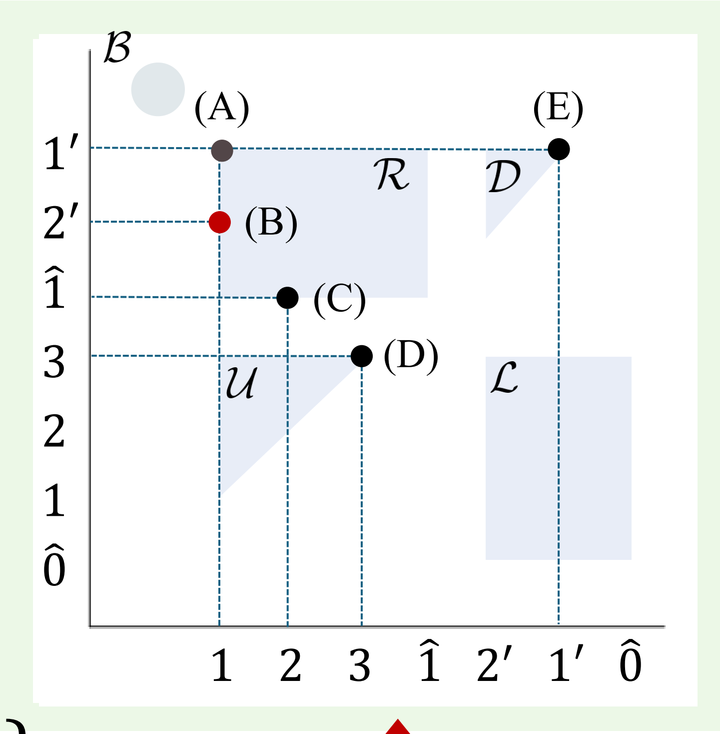
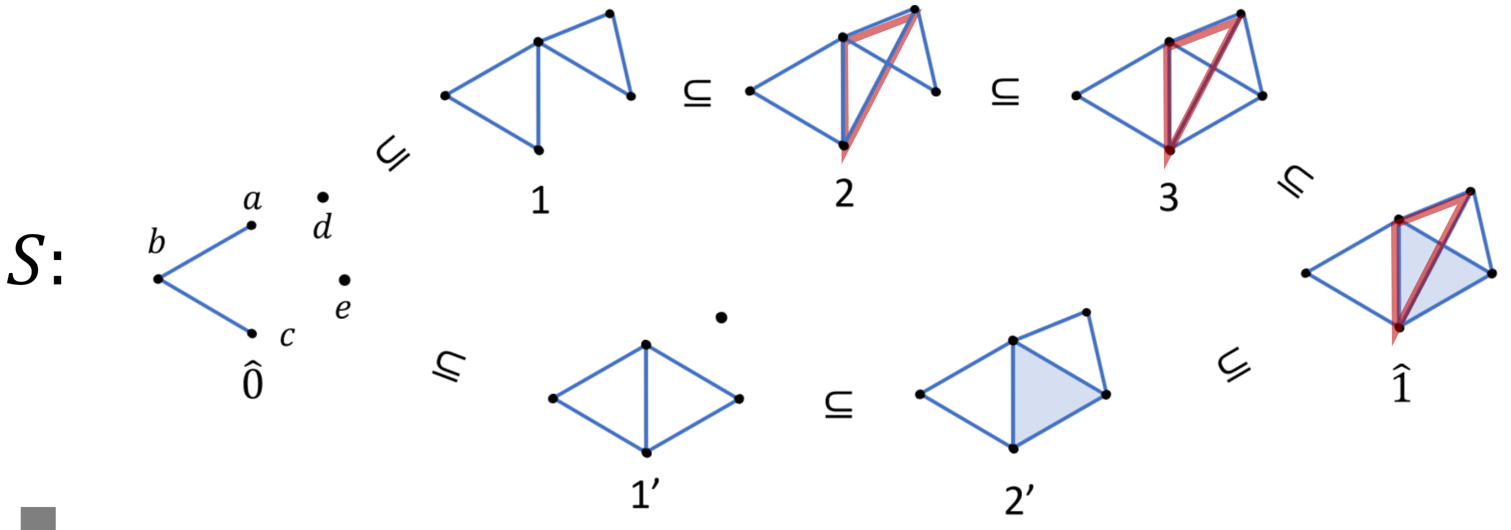
(A) (B) (C) (D) (E)



Bipath PD



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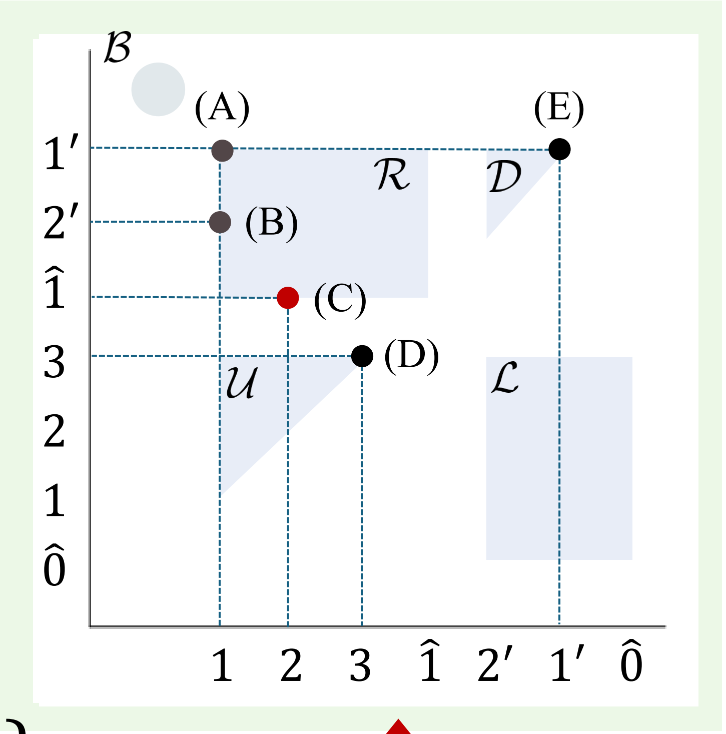
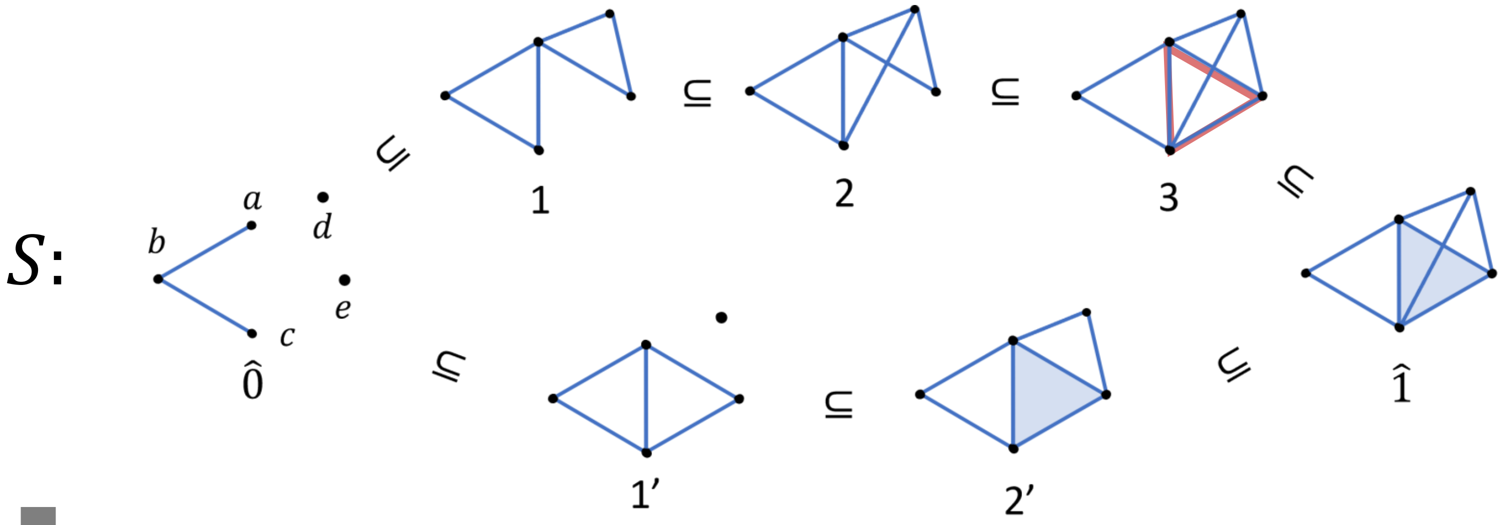
$$= \{\underset{(A)}{\langle 1,1'\rangle}, \underset{(B)}{\langle 1,2'\rangle}, \underset{(C)}{\langle 2,\hat{1}\rangle}, \underset{(D)}{\langle 3,3\rangle}, \underset{(E)}{\langle 1',1'\rangle}\}.$$



Bipath PD



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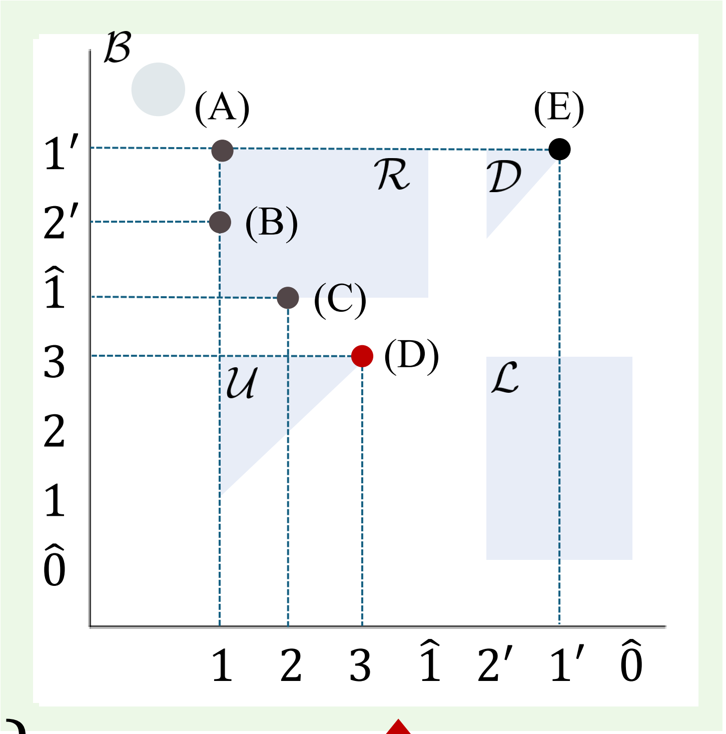
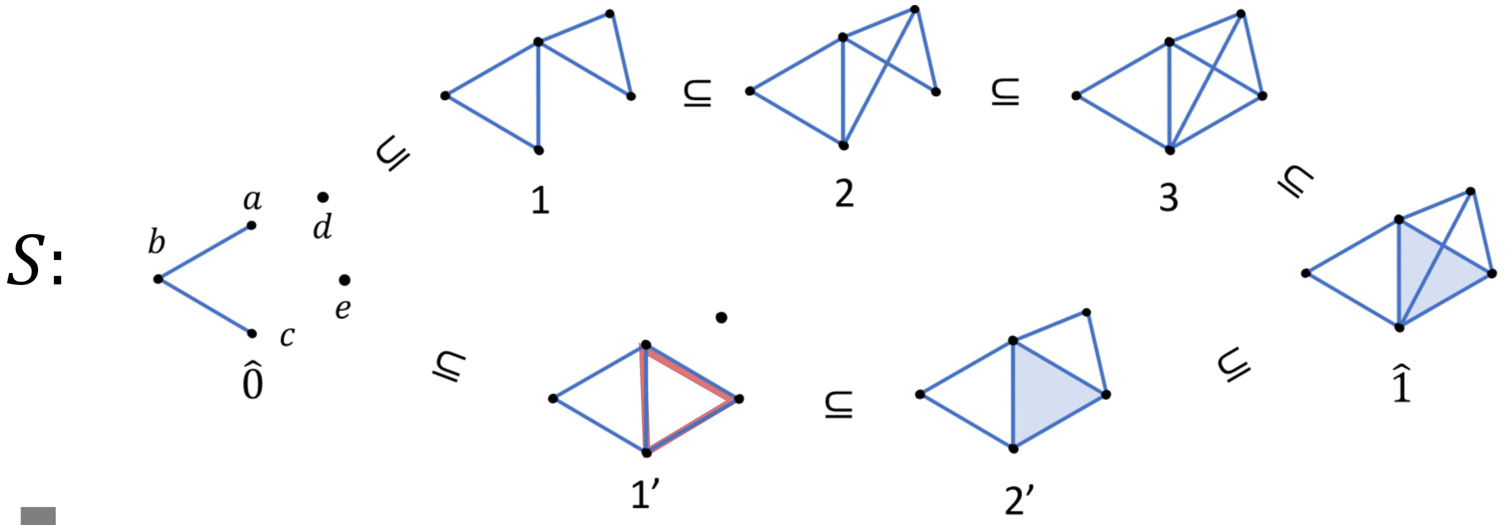
(A) (B) (C) (D) (E)



Bipath PD



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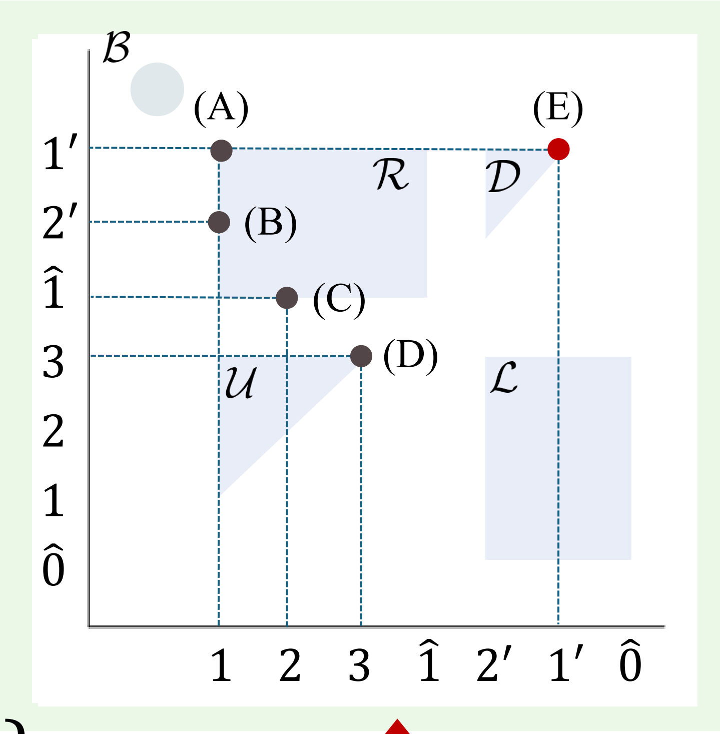
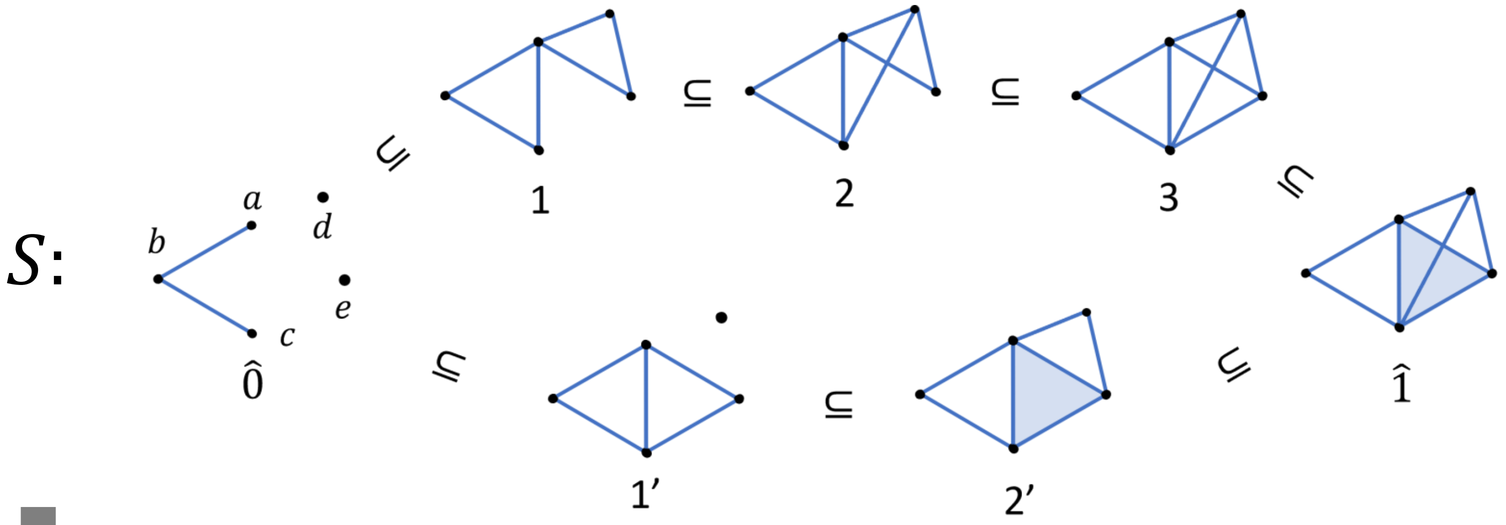
(A) (B) (C) (D) (E)



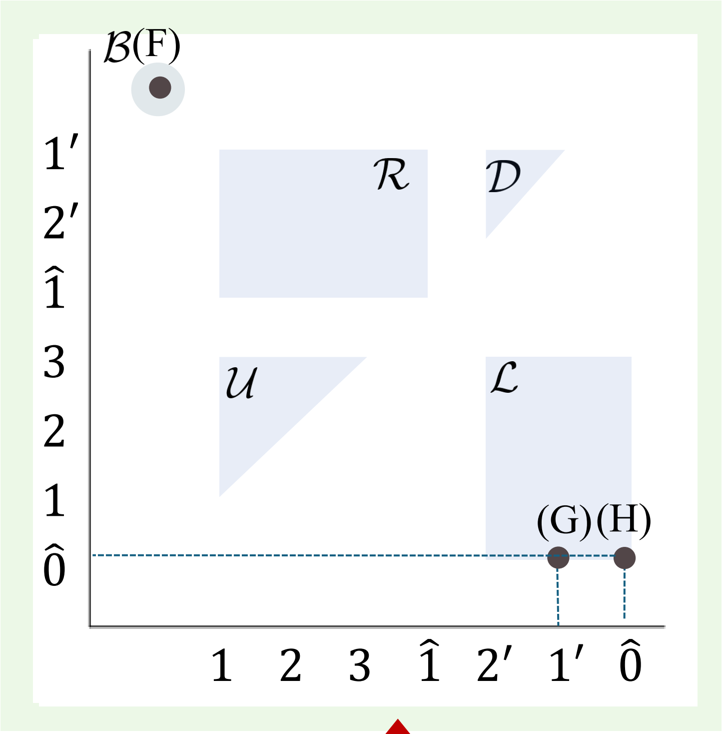
Bipath PD



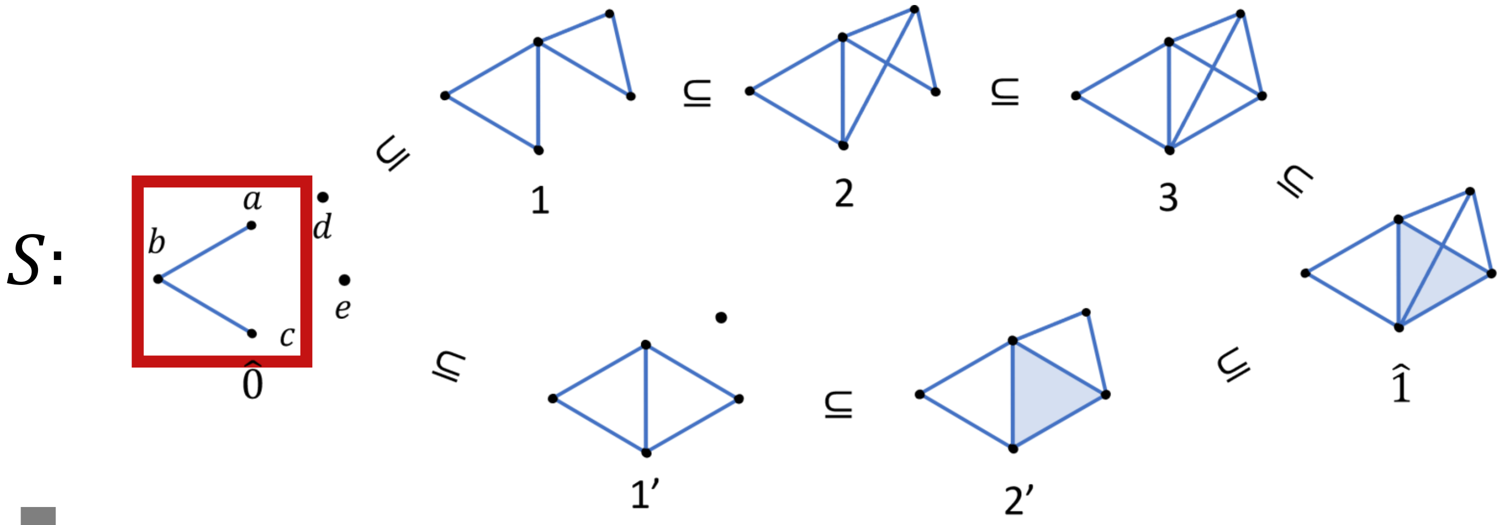
- Compute bipath PD of $H_0(S; k = \mathbb{F}_2)$.
- Compare upper and lower filtration.

$$\mathcal{B}(H_0(S;k)) = \{B_{3,2}, \{1', \hat{0}\}, \{\hat{0}\}\}$$

$$= \{\underset{\text{(F)}}{B_{3,2}}, \underset{\text{(G)}}{\langle \hat{0}, \hat{0} \rangle}, \underset{\text{(H)}}{\langle 1', \hat{0} \rangle}\}$$



Bipath PD



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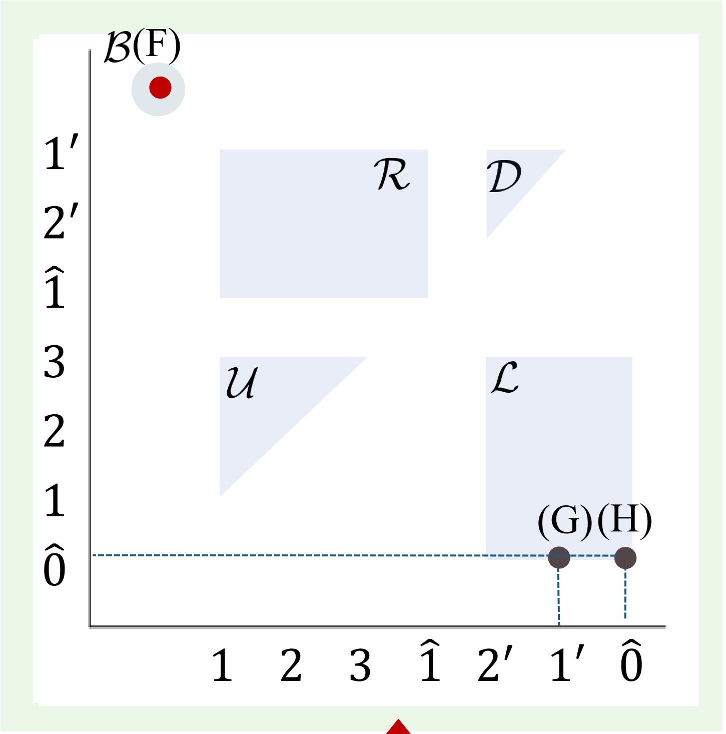
- Compare upper and lower filtration.

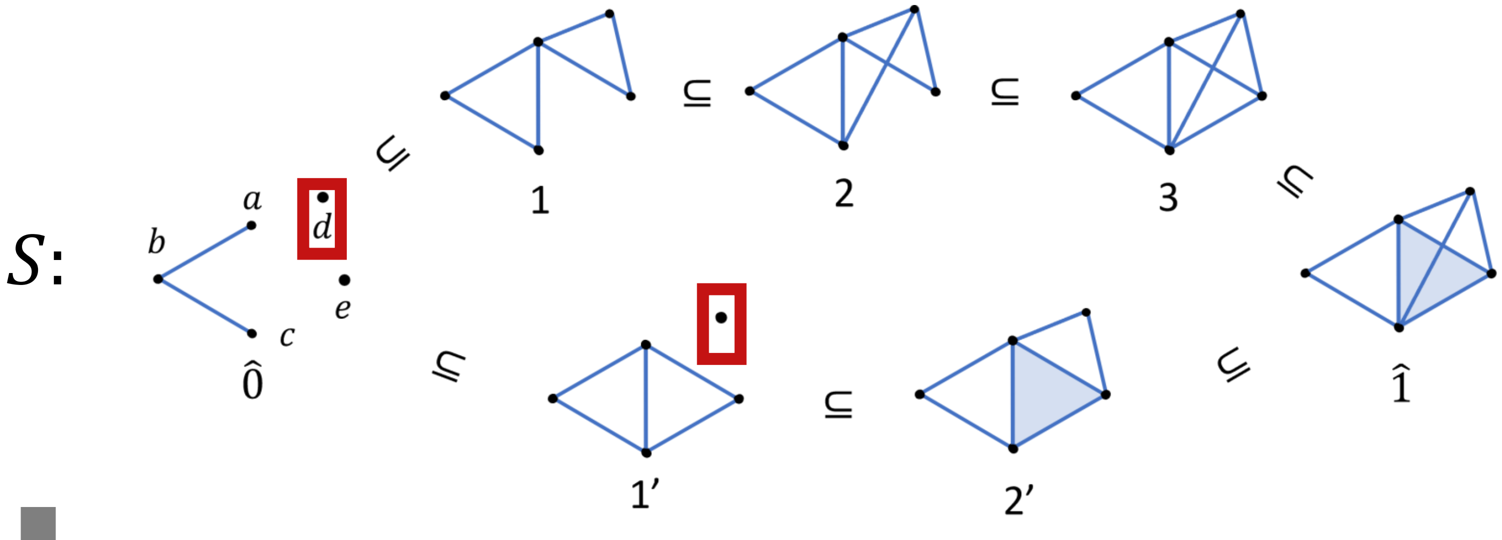
$$\mathcal{B}(H_0(S;k)) = \{B_{3,2}, \{1', \hat{0}\}, \{\hat{0}\}\}$$

$$= \{B_{3,2}, \langle \hat{0}, \hat{0} \rangle, \langle 1', \hat{0} \rangle\}$$

(F) (G) (H)



Bipath PD



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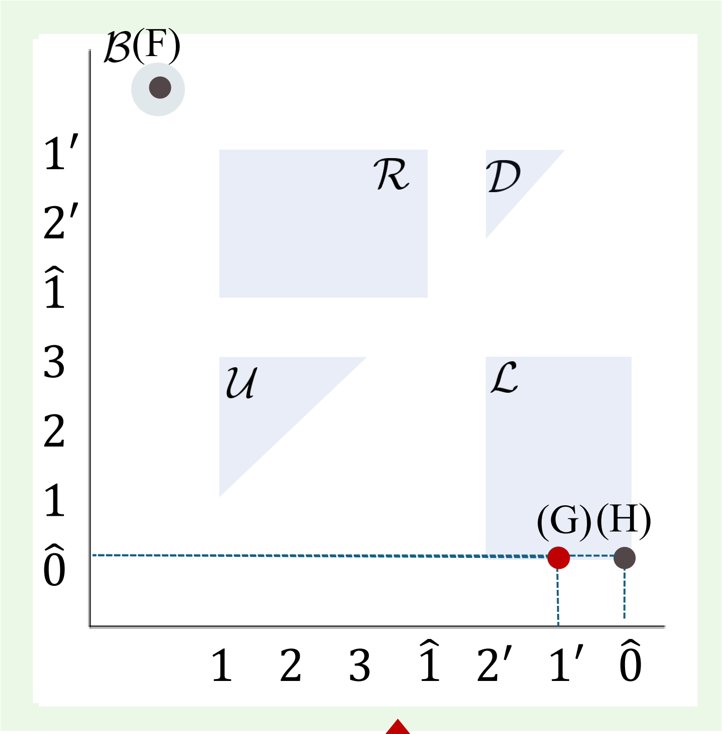
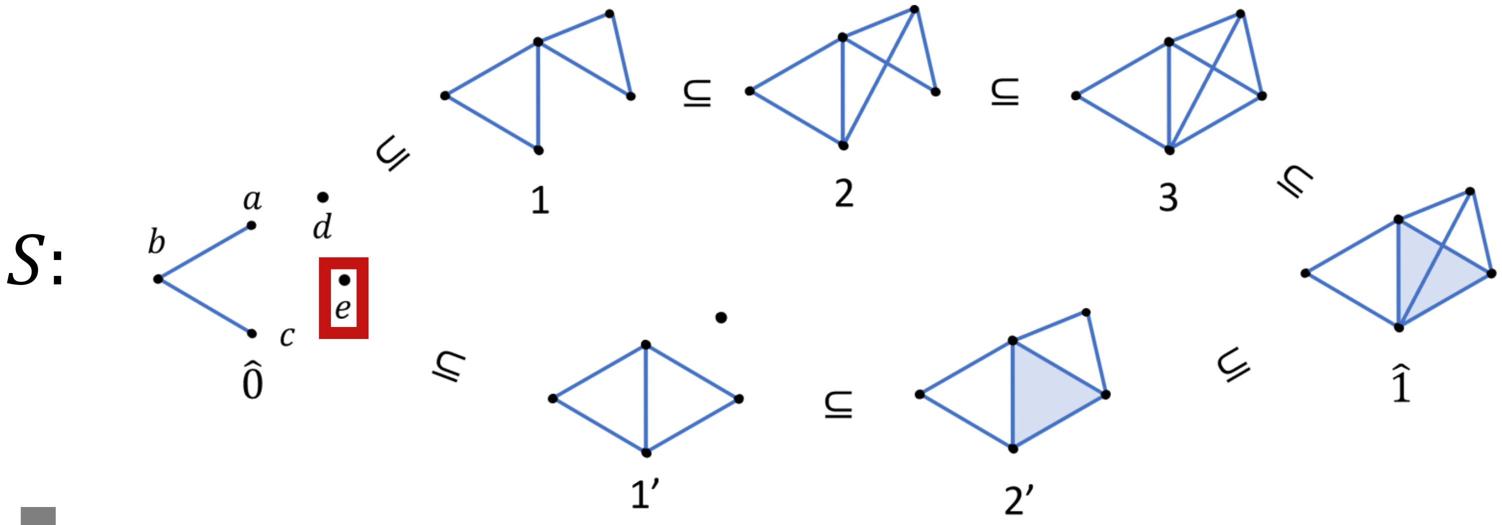


Bipath PD



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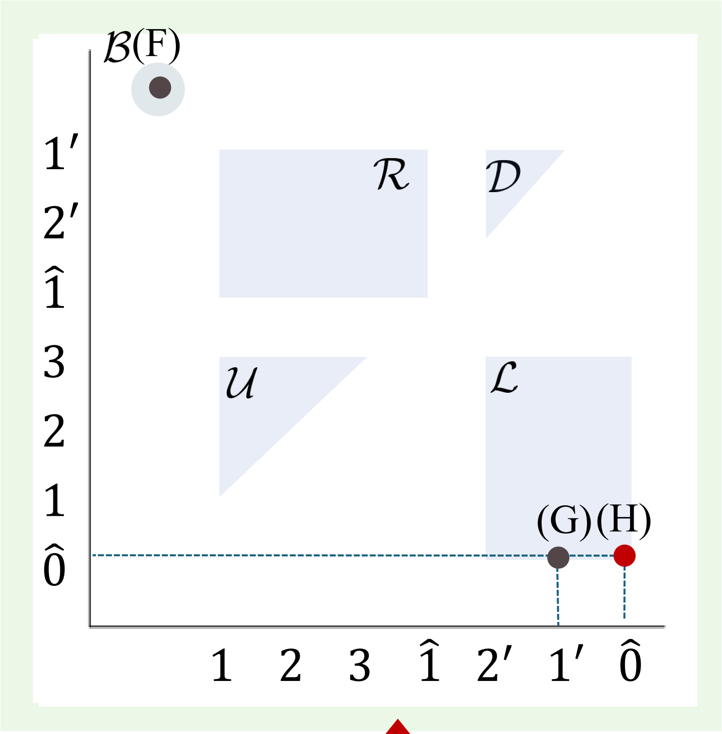
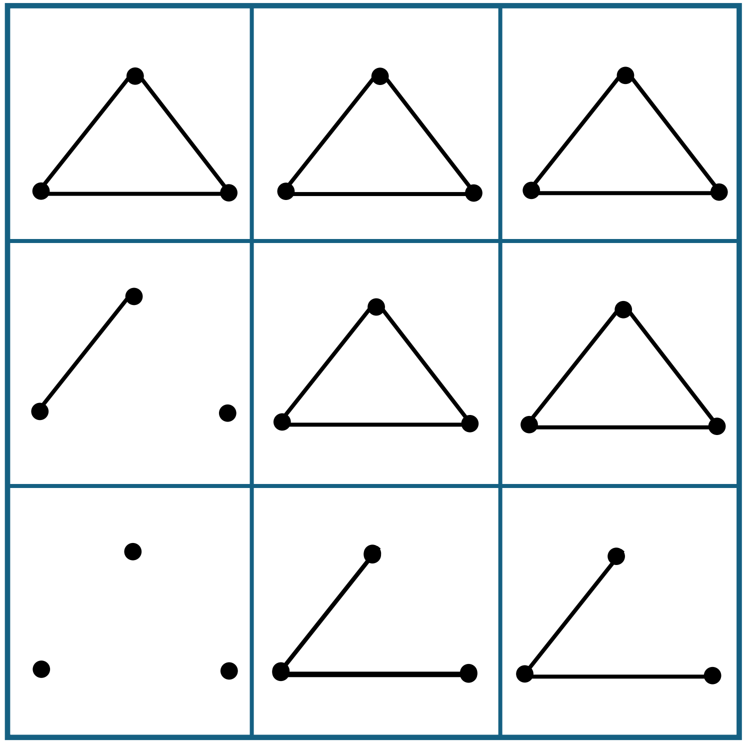
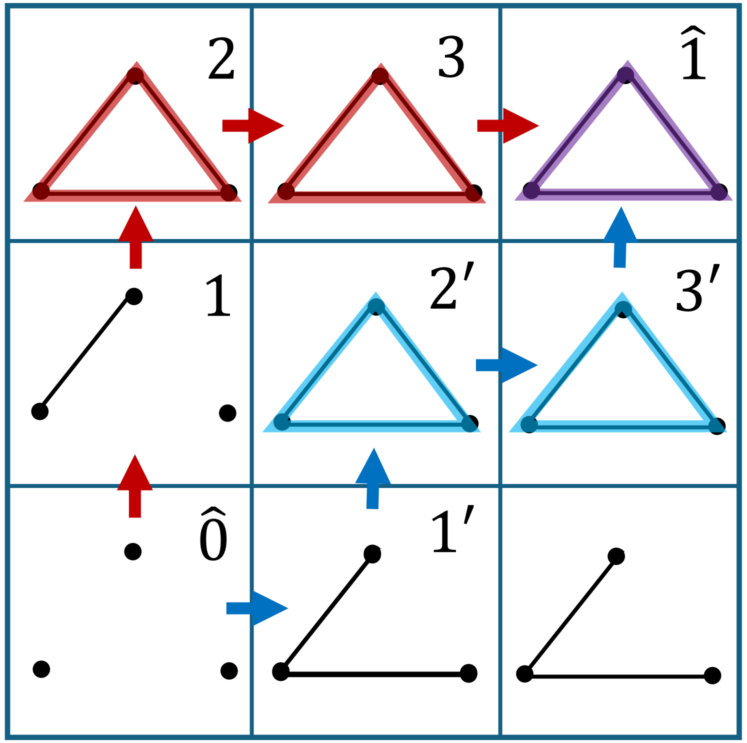


Bipath PD



Bipath PD

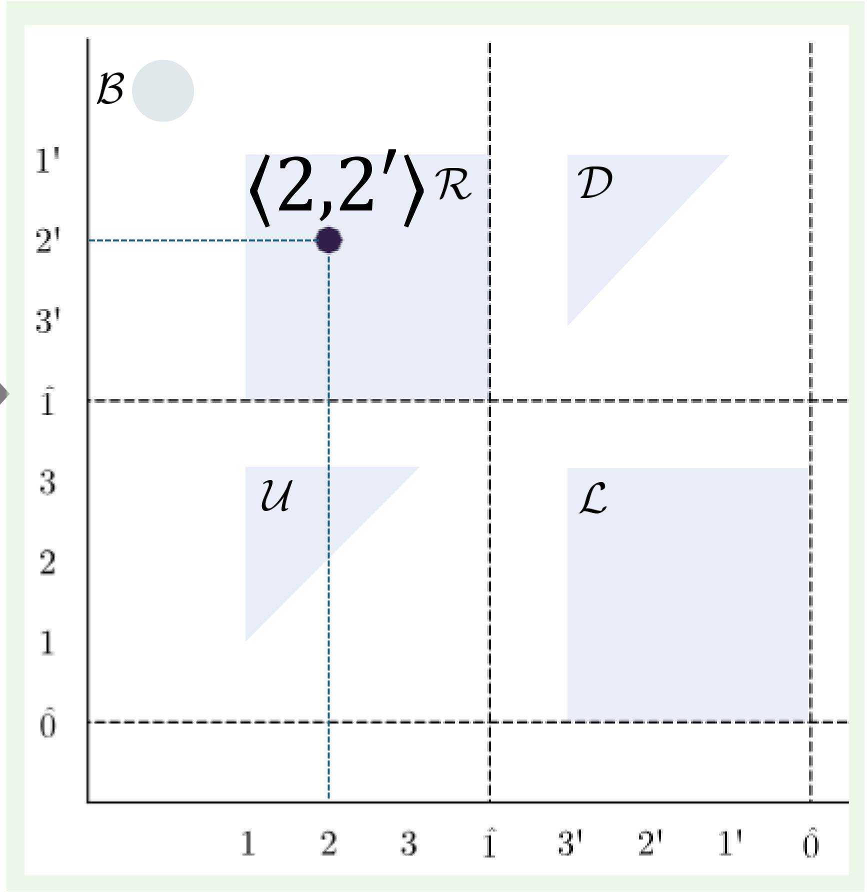


:Upper path

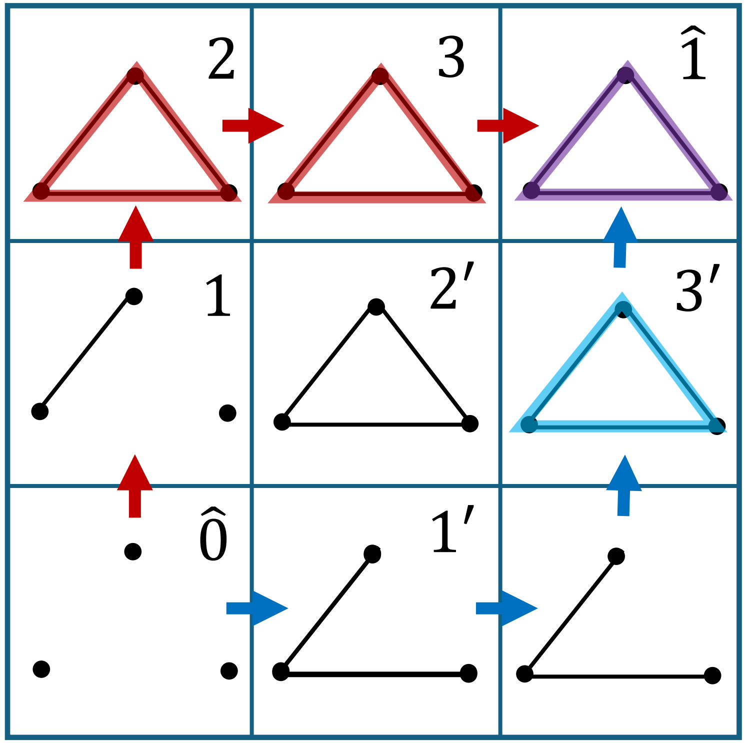
:Lower path

Get bipath PH.

$(k = \mathbb{F}_2)$



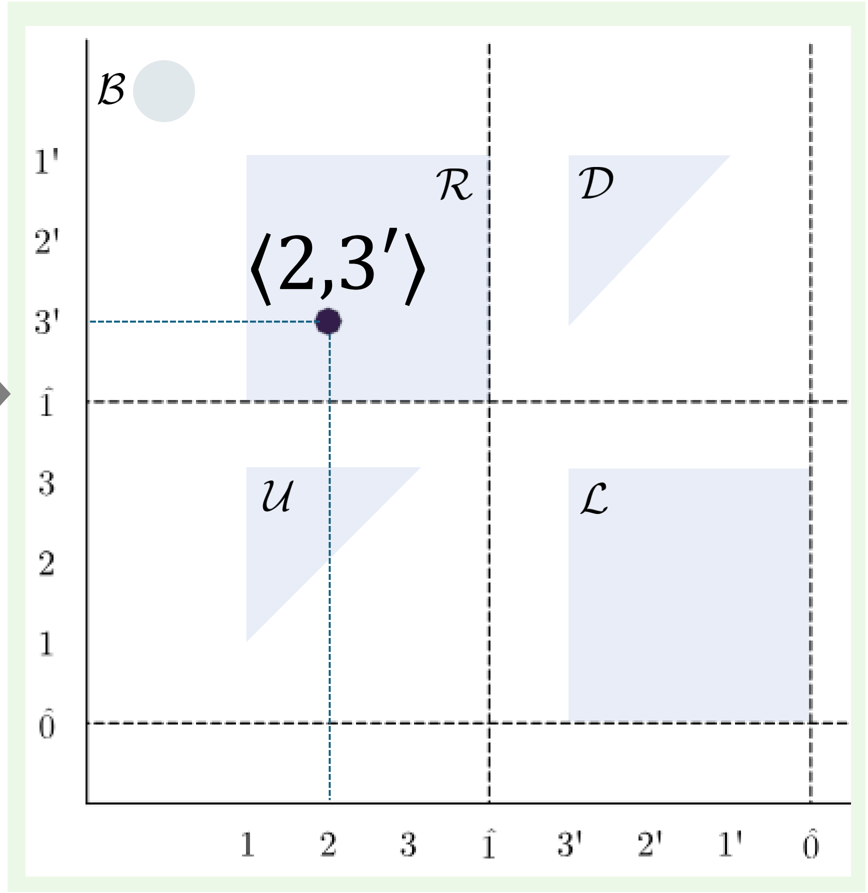
Bipath PD



:Upper path

:Lower path

Get bipath PH.
$(k = \mathbb{F}_2)$



An Algorithm for Computing Bipath PD

We gave a computational method for bipath PD.

Theorem [Aoki-Escolar-T, 24]

A computational complexity of our algorithm for computing bipath PD is

$$O(Z + m^3),$$

where

- $O(Z)$ is computational complexity of standard algorithm for PH, and
- m is the maximal size of invertible matrices Λ and Γ (later).

*(2 times of standard algorithm for PH) + (computation on Λ and Γ).

Bipath PD can be computed without much more effort than standard algorithm.

An algorithm for bipath PD

Input: S bipath filtration of simplicial complex

$$S: \quad S_{\hat{0}} \begin{array}{l} \nearrow \; S_1 \to \cdots \to s_n \; \searrow \\ \searrow \; S_{1'} \to \cdots \to s_{m'} \; \nearrow \end{array} S_{\hat{1}}$$

<u>Step 0</u>. Separate a bipath filtration S into S_U and S_D.

$$\begin{array}{l} S_U: \; S_{\hat{0}} \to S_1 \to \cdots \to s_n \to S_{\hat{1}} \\ \qquad\;\; \parallel \qquad\qquad\qquad\qquad\qquad\quad \parallel \\ S_D: \; S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to S_{\hat{1}} \end{array}$$

An algorithm for bipath PD

Input: S bipath filtration of simplicial complex

$$S\colon\quad S_{\hat{0}} \begin{array}{l} \nearrow\; S_1 \to \cdots \to s_n \;\searrow \\ \searrow\; S_{1'} \to \cdots \to s_{m'} \;\nearrow \end{array} S_{\hat{1}}$$

Step 1. Get intervals of S_U and S_D by standard algorithm.

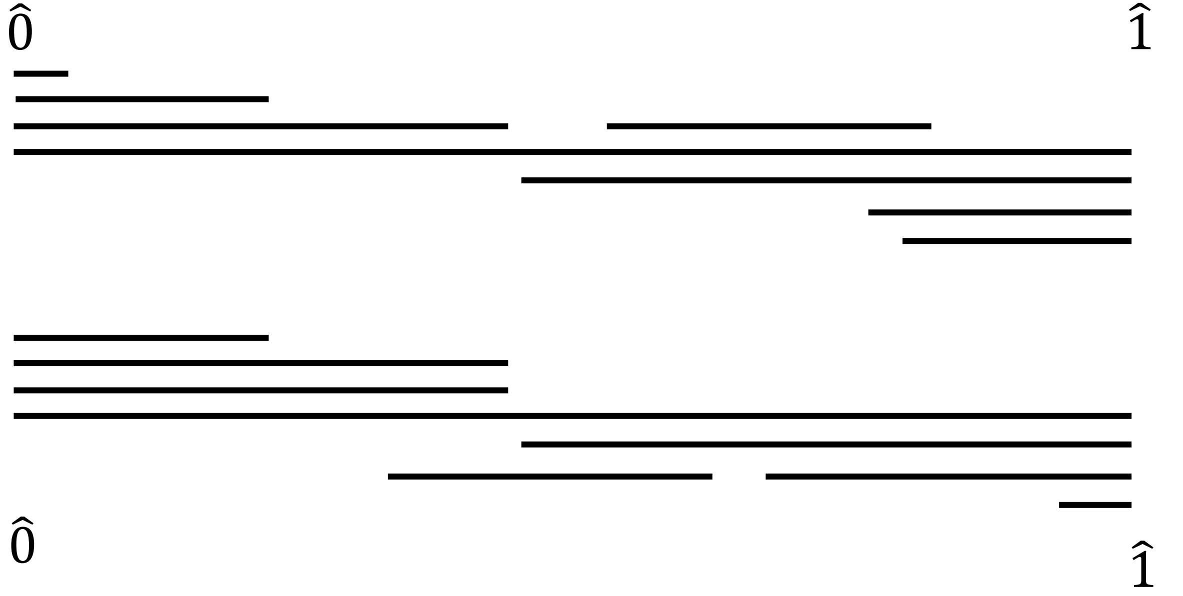
$$S_U\colon\; S_{\hat{0}} \to S_1 \to \cdots \to s_n \to S_{\hat{1}}$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$S_D\colon\; S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to S_{\hat{1}}$$

$H_q(S_U) \cong X =$

$H_q(S_D) \cong Y =$



An algorithm for bipath PD

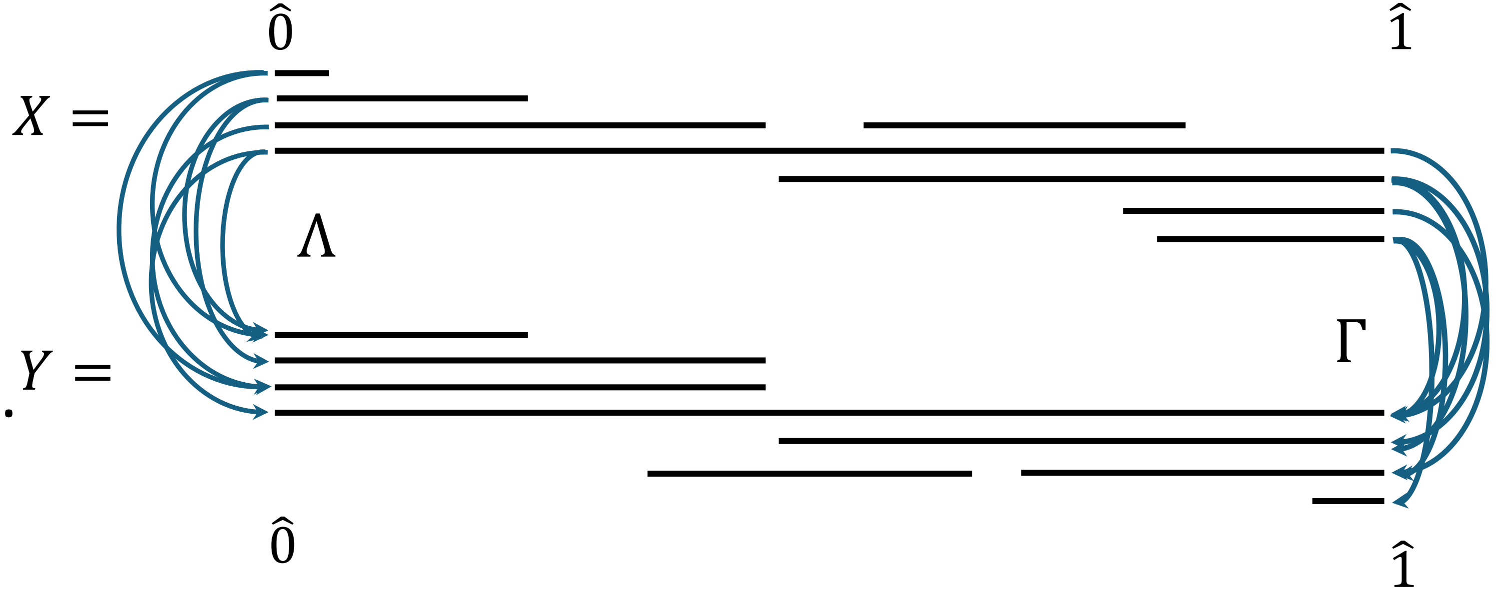
Input: S bipath filtration of simplicial complex

$$S:\quad S_{\hat{0}} \begin{array}{l} \nearrow \; S_1 \to \cdots \to s_n \; \searrow \\ \searrow \; S_{1'} \to \cdots \to s_{m'} \; \nearrow \end{array} S_{\hat{1}}$$

<u>Step 2</u>. Compute change-of-basis matrices

$$\Lambda: X_{\hat{0}} \to Y_{\hat{0}}$$
$$\Gamma: X_{\hat{1}} \to Y_{\hat{1}}.$$

$$S_U:\ S_{\hat{0}} \to S_1 \to \cdots \to s_n \to S_{\hat{1}}$$
$$\qquad \| \qquad\qquad\qquad\qquad\qquad \|$$
$$S_D:\ S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to S_{\hat{1}}$$



- The size of matrices Λ and Γ is smaller than #intervals.

-> Usually smaller than #simplices.

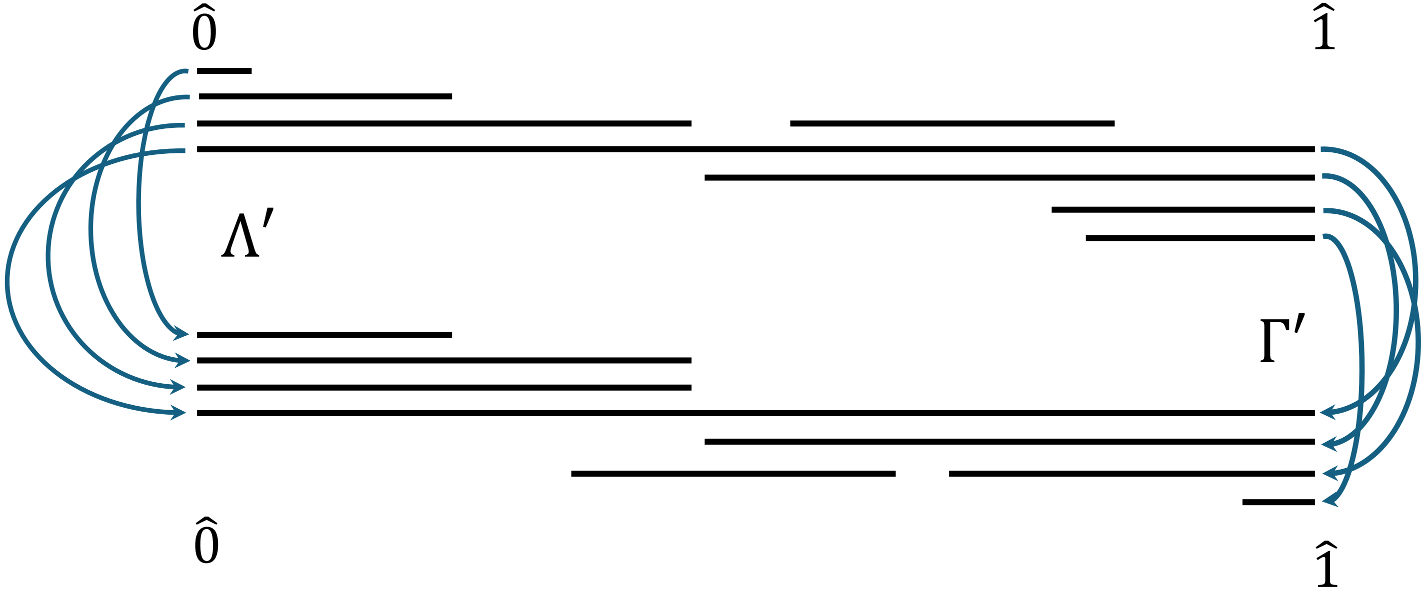
An algorithm for bipath PD

Input: S bipath filtration of simplicial complex

$$S: \quad S_{\hat{0}} \begin{array}{l} \nearrow S_1 \to \cdots \to s_n \searrow \\ \searrow S_{1'} \to \cdots \to s_{m'} \nearrow \end{array} S_{\hat{1}}$$

Step 3. Reduce Λ and Γ to permutation matrices with preserving upper and lower interval decompositions.

$$S_U: \ S_{\hat{0}} \to S_1 \to \cdots \to s_n \to S_{\hat{1}}$$
$$\parallel \qquad\qquad\qquad\qquad \parallel$$
$$S_D: \ S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to S_{\hat{1}}$$



An algorithm for bipath PD

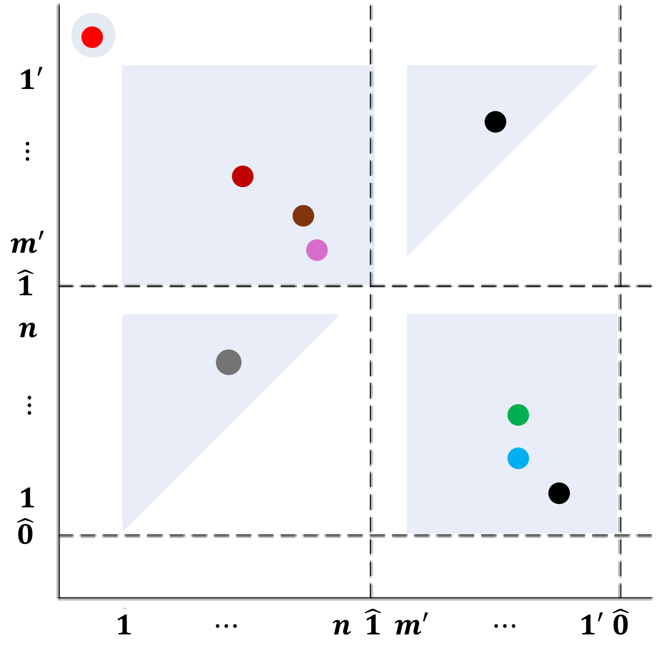
Input: S bipath filtration of simplicial complex

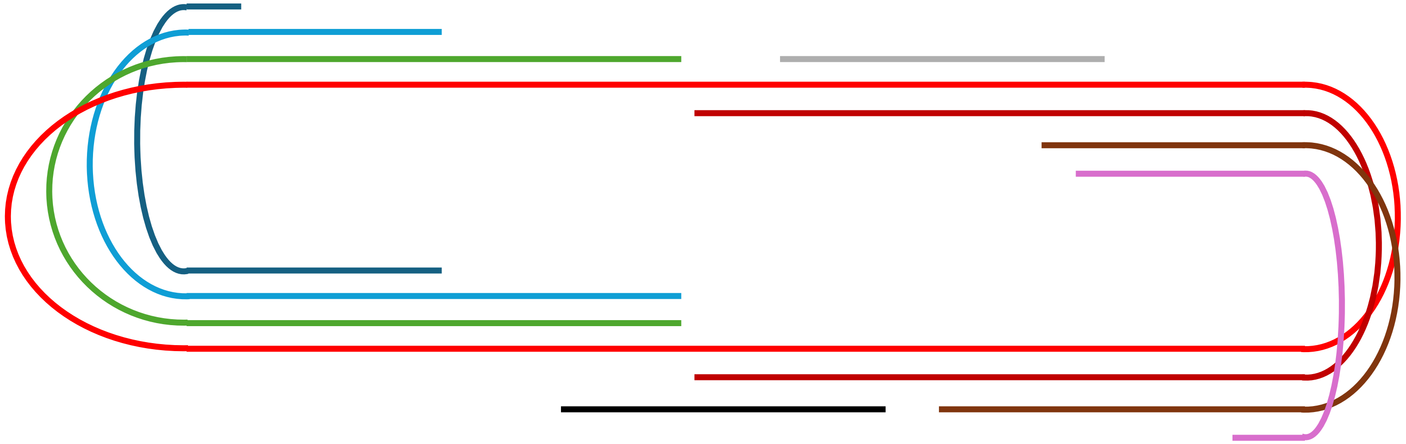
$$
\begin{array}{cccccc}
 & & S_1 \to \cdots \to s_n & & \\
S: & S_{\hat{0}} \nearrow & & \searrow & S_{\hat{1}} \\
 & \searrow & S_{1'} \to \cdots \to s_{m'} & \nearrow &
\end{array}
$$

<u>Step 4</u>. Connect upper and lower intervals, and get intervals.

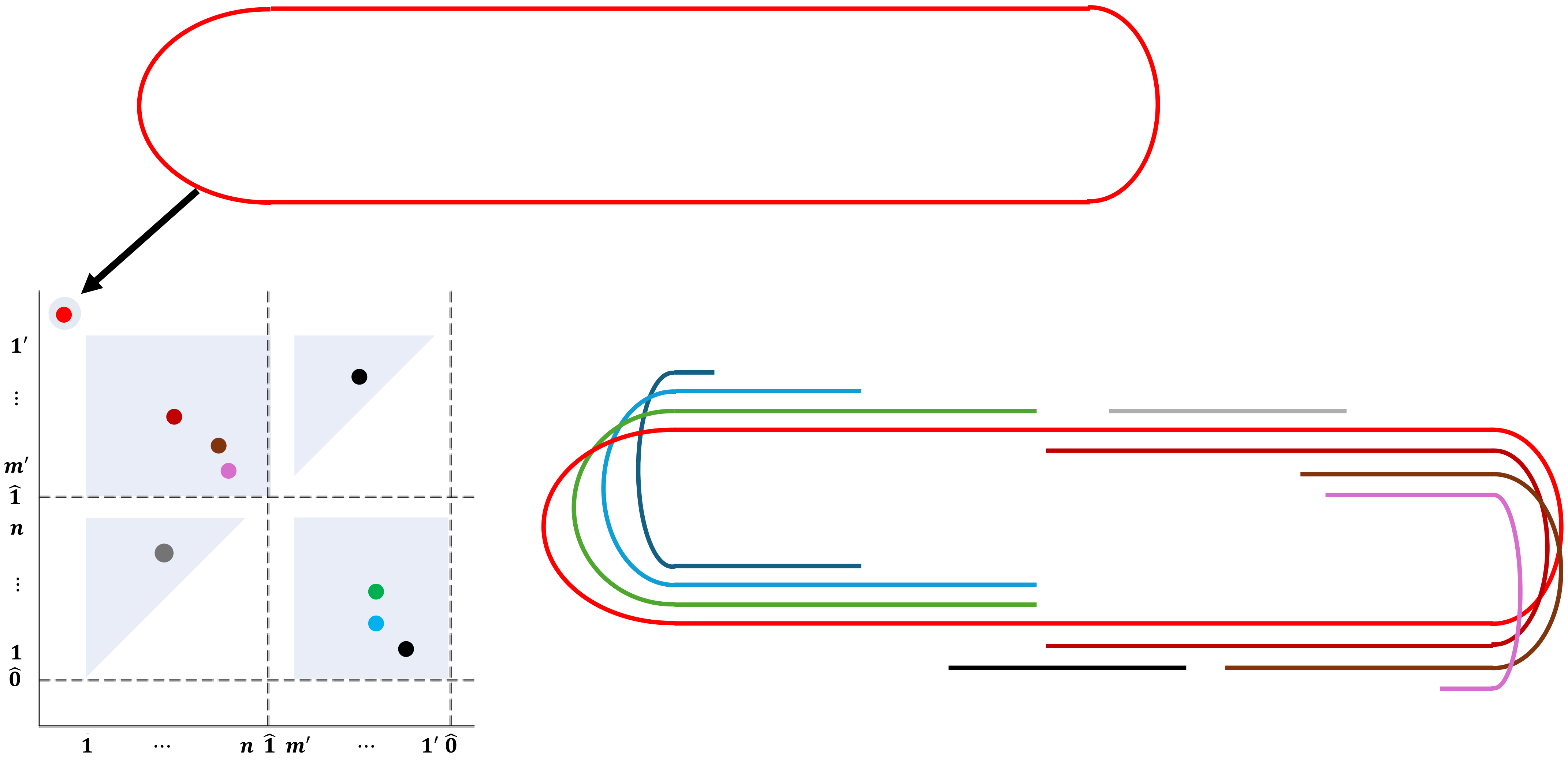
$$
\begin{array}{l}
S_U:\ S_{\hat{0}} \to S_1 \to \cdots \to s_n \to S_{\hat{1}} \\
\qquad \parallel \qquad\qquad\qquad\qquad\quad \parallel \\
S_D:\ S_{\hat{0}} \to S_{1'} \to \cdots \to s_{m'} \to S_{\hat{1}}
\end{array}
$$

Output:



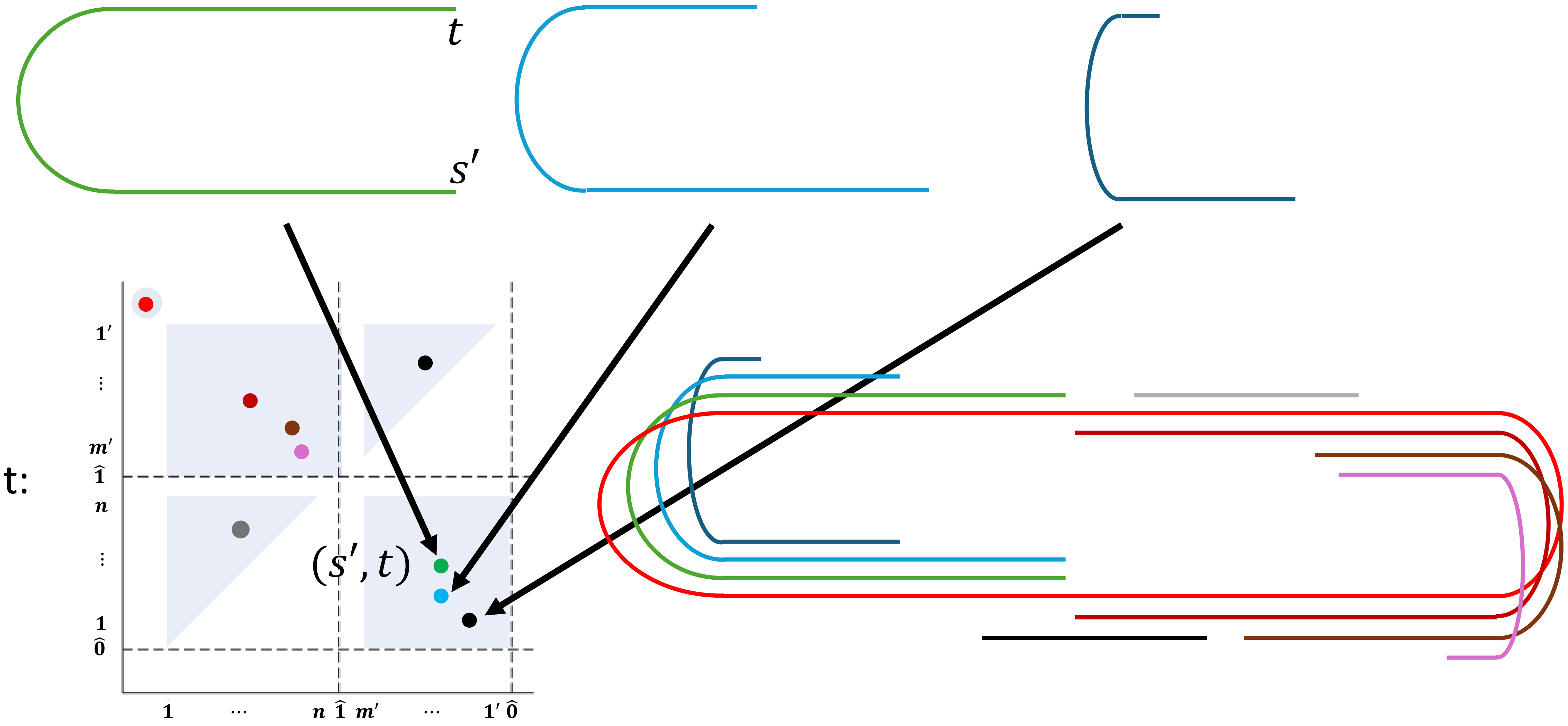


An algorithm for bipath PD



An algorithm for bipath PD

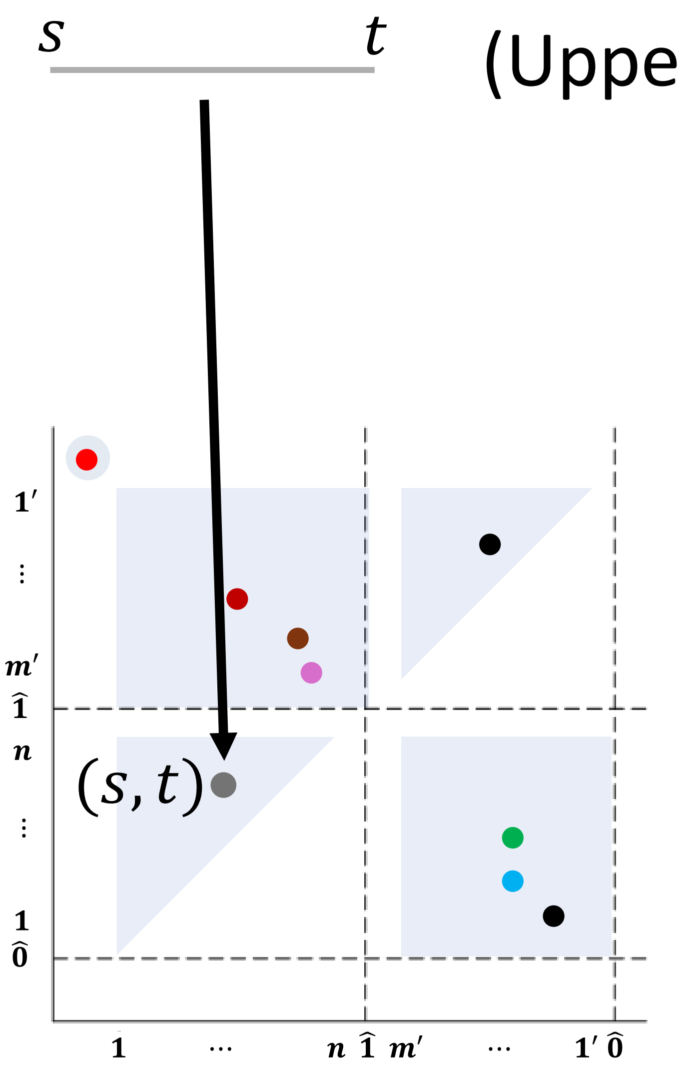
Output:

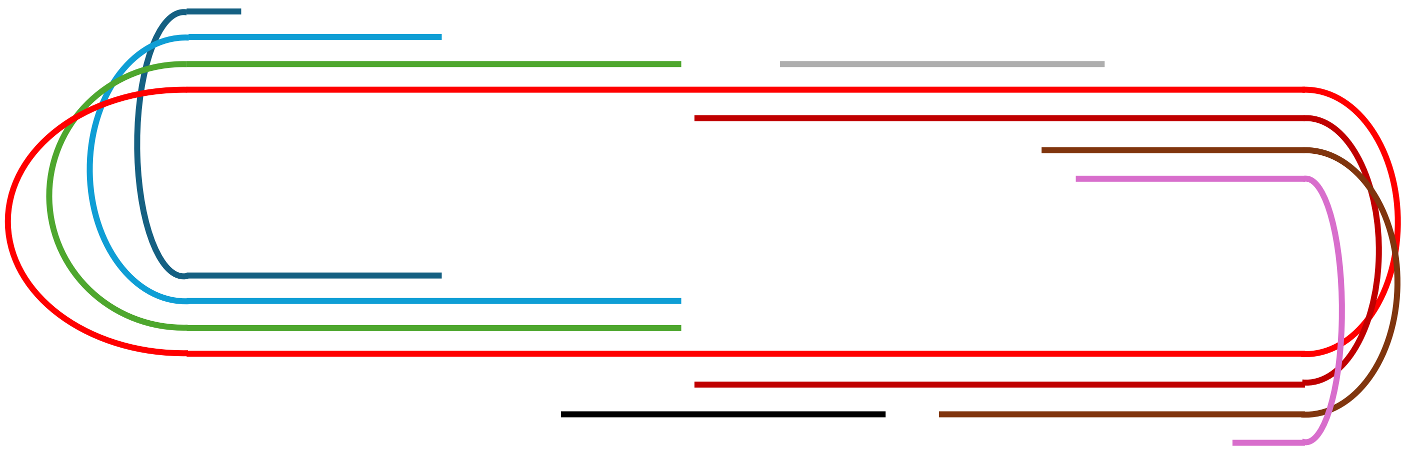


An algorithm for bipath PD

(Upper)

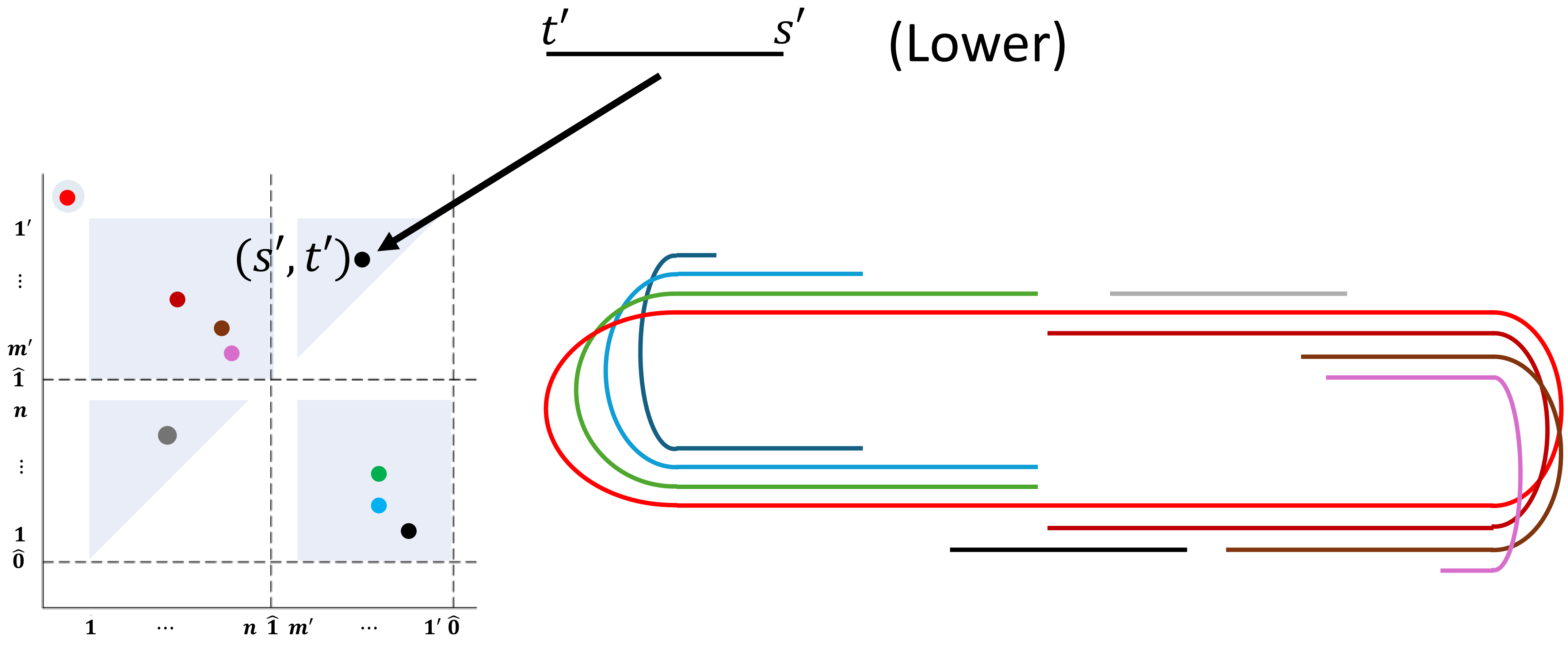
Output:





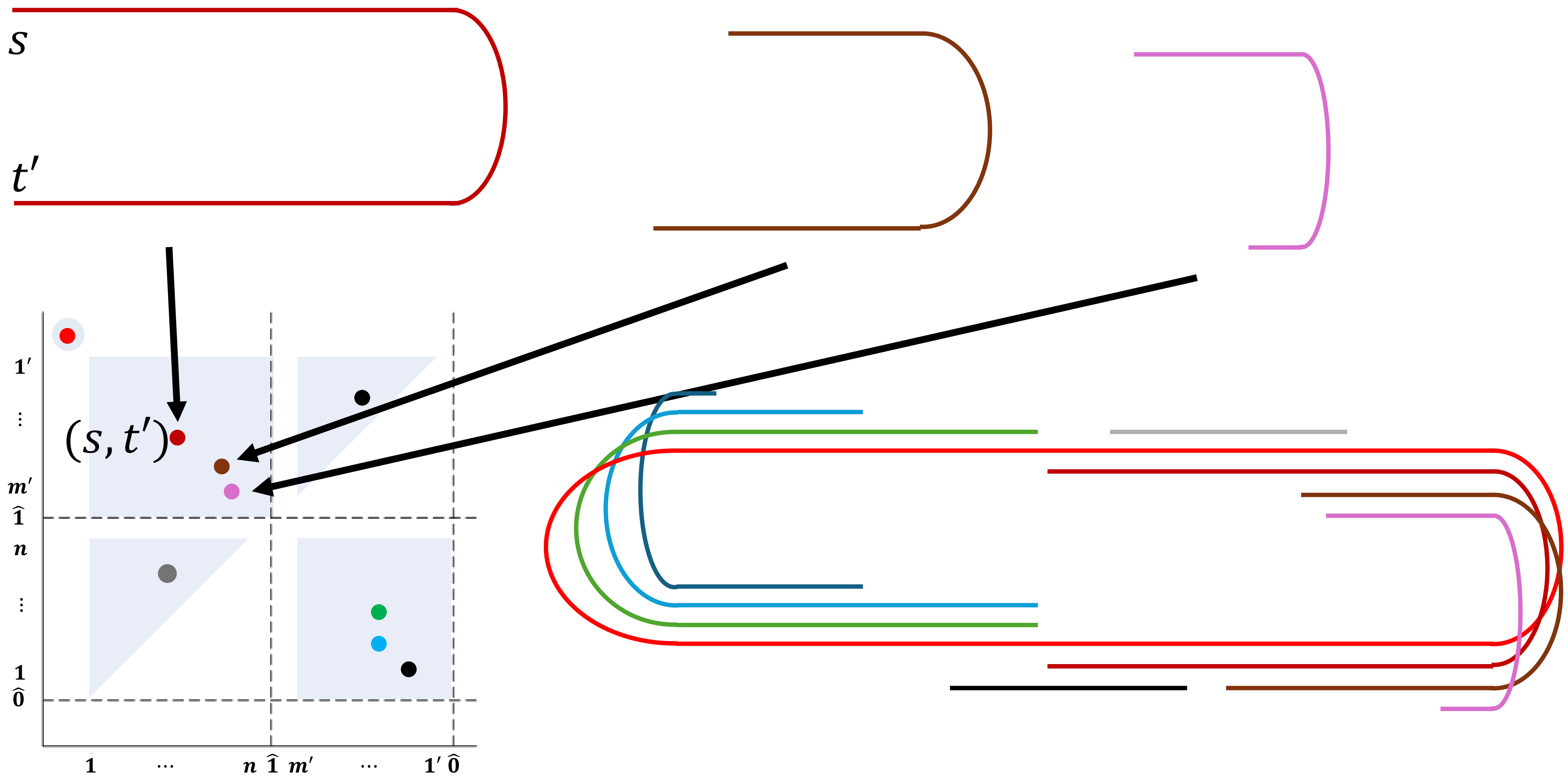
An algorithm for bipath PD

Output:



An algorithm for bipath PD

Output:



An algorithm for bipath PD

Remark.

Our algorithm is implemented.

https://github.com/ShunsukeTada1357/Bipathposets.

Note.

・We get a bipath PD by 2 times of standard algorithm for PH and matrix operations on Λ and Γ (whose size depend on intervals).

->Bipath PD can be computed without much more effort than standard algorithm.

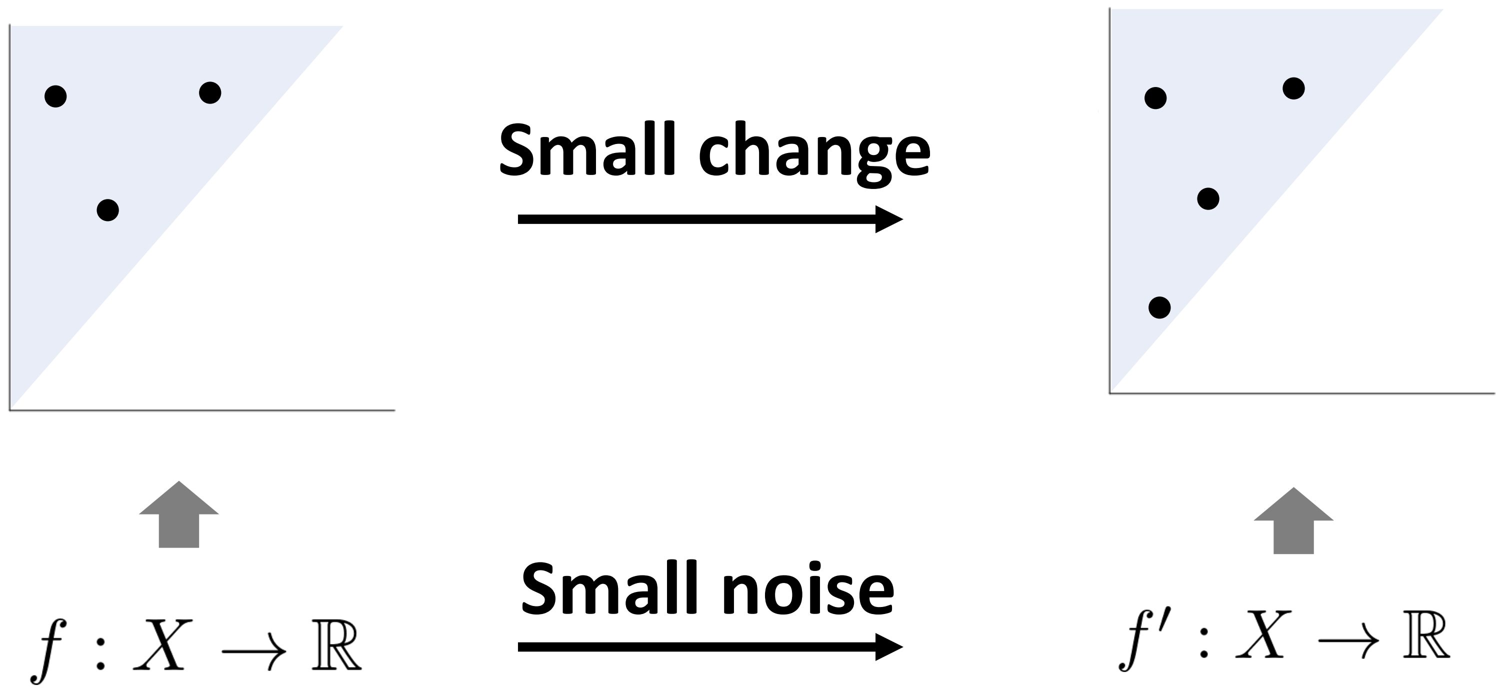
・Mathematical foundation for our algorithm is in our paper “Bipath Persistence” (arXiv:2404.02536).

“Stability of bipath persistence diagram”

(In preparation)

Background: Stability theorem for standard PH

Small noise of data implies small change in PD.



Background: Stability theorem for standard PH

Stability theorem (see [Frédéric Chazal, et al. '2009] for example)

Let f and g be a real-valued function on top. sp. X. Then, we have

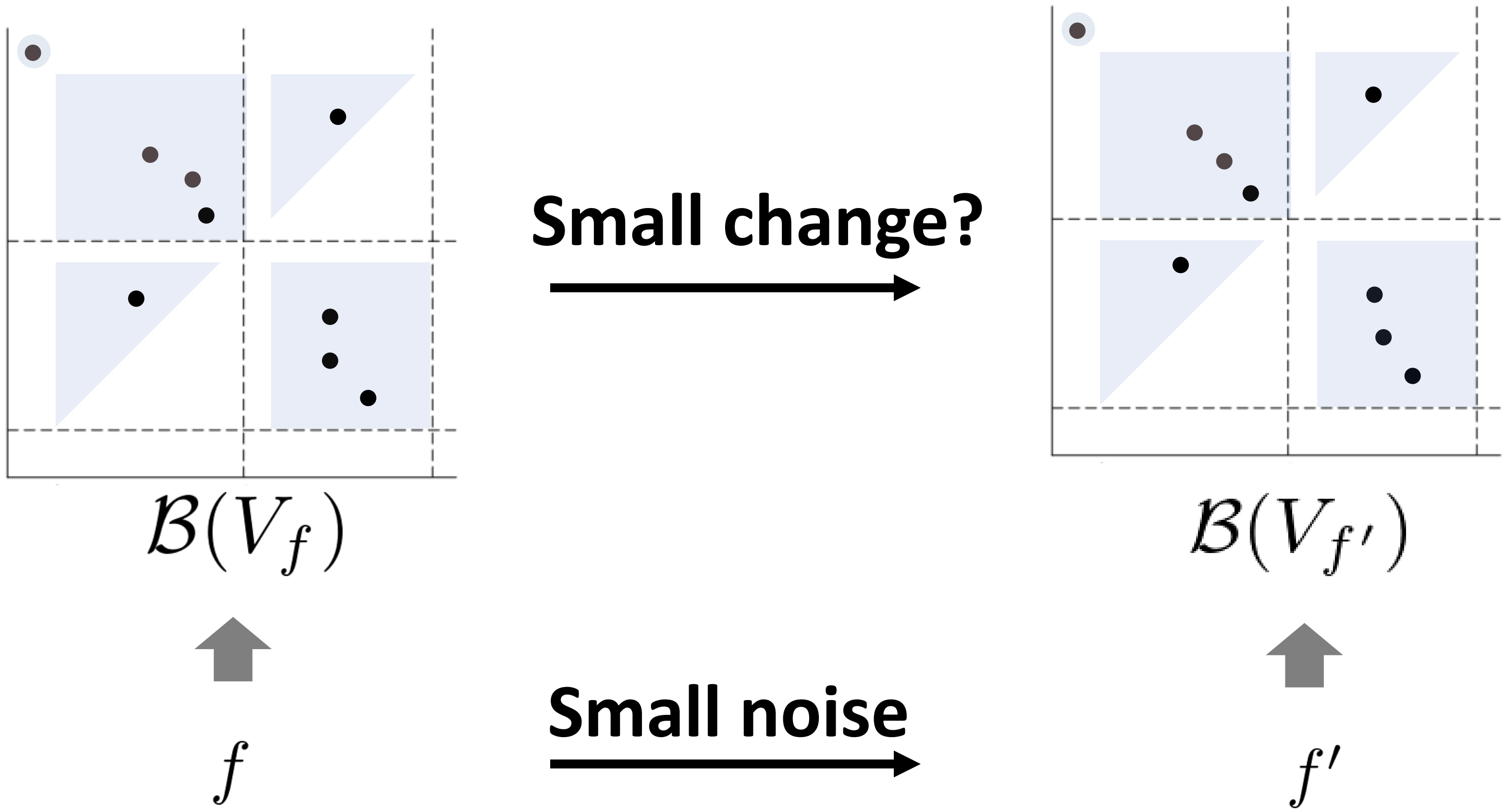
$$d_{\mathrm{B}}(\mathcal{B}(V_f), \mathcal{B}(V_g)) \leq ||f - g||_{\infty}.$$

- $\mathcal{B}(V_f)$: The standard PD of PH by sublevelset filtration of f.
- $\mathcal{B}(V_g)$: The standard PD of PH by sublevelset filtration of g.
- d_{B}: Bottleneck distance between PDs.
- $||f - g||_{\infty} := \sup_{x \in X} |f(x) - g(x)|$.

・David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Stability of persistence diagrams. Discrete& Computational Geometry, 37:103–120, 2007.
・Frédéric Chazal, David Cohen-Steiner, Marc Glisse, Leonidas J Guibas, and Steve Y Oudot. Proximityof persistence modules and their diagrams. In Proceedings of the twenty-fifth annual symposium onComputational geometry, pages 237–246, 2009.

Stability theorem for bipath PD

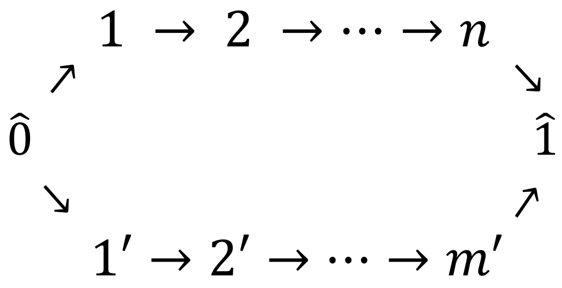
Next, we discuss stability theorem for bipath PDs.



Stability theorem for bipath PD: Setting

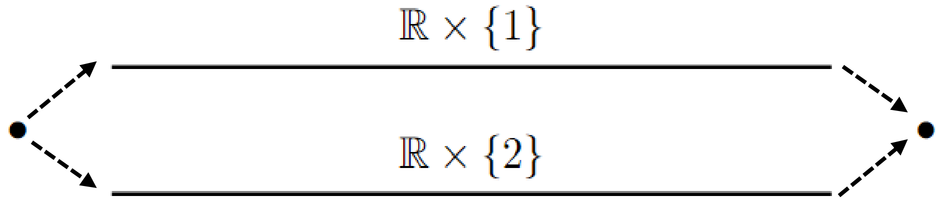
We consider continuous version of a bipath poset B to discuss stability.
(We have a natural embedding $\iota\colon B_{n,m} \hookrightarrow B$)

$B_{n,m}$:



$$\{1, \ldots, n\} \sqcup \{1', \ldots, m'\} \sqcup \{\hat{0}, \hat{1}\}$$

B: $-\infty$



$+\infty$

$$(\mathbb{R} \times \{1\}) \sqcup (\mathbb{R} \times \{2\}) \sqcup \{-\infty, +\infty\}$$

Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath function)

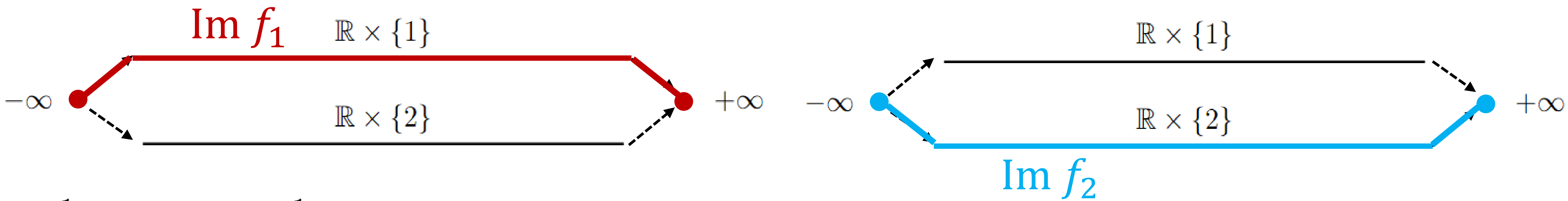
A *bipath function* f on top. sp. X is a pair of B-valued functions f_1 and f_2 on X such that

$$\operatorname{Im} f_i \subseteq (\mathbb{R} \times \{i\}) \sqcup \{\pm\infty\} \subseteq B$$

and

$$f_1^{-1}(-\infty) = f_2^{-1}(-\infty).$$

We denote by $f : X \to B$ a bipath function on X.



*$f_1^{-1}(-\infty) = f_2^{-1}(-\infty)$ is needed to define a bipath sublevelset filtration.

Stability theorem for bipath PD: Bipath sublevelset filtration

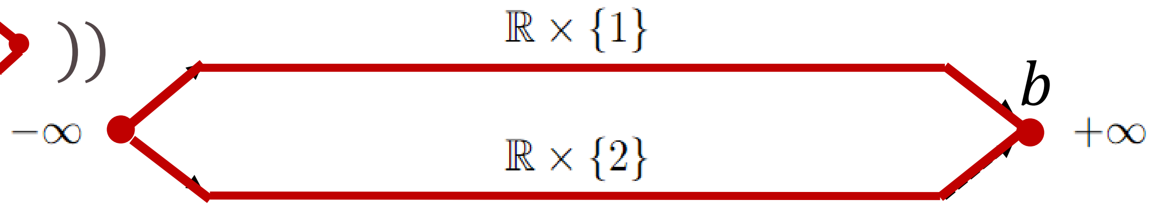
Definition (Bipath sublevelset filtration)

Let f be a bipath function on a top. sp. X. For any b in B, let

$$(f \leq b) := \begin{cases} X & \text{if } b = +\infty \\ f_1^{-1}(\{-\infty\}) & \text{if } b = -\infty \\ f_1^{-1}([-\infty, r] \times \{1\}) & \text{if } b = (r, 1) \\ f_2^{-1}([-\infty, r] \times \{2\}) & \text{if } b = (r, 2) \end{cases}$$

Then, they give a functor $(f \leq \cdot) : B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

$(f \leq +\infty) = X(= f_1^{-1}($  $))$



Stability theorem for bipath PD: Bipath sublevelset filtration

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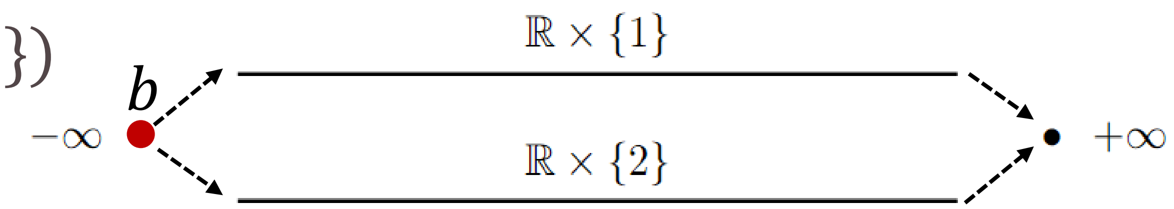
$(f \leq -\infty) = f_1^{-1}(\bullet = \{-\infty\})$



Stability theorem for bipath PD: Bipath sublevelset filtration

Definition (Bipath sublevelset filtration)

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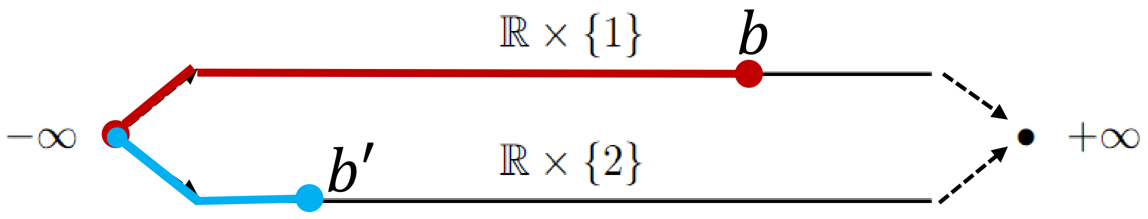
Then, they give a functor $(f \leq \cdot) : B \to \mathrm{Top}$. We call it *bipath sublevelset filtration*.

$(f \leq b) = f_1^{-1}($  $)$

$(f \leq b') = f_2^{-1}($  $)$

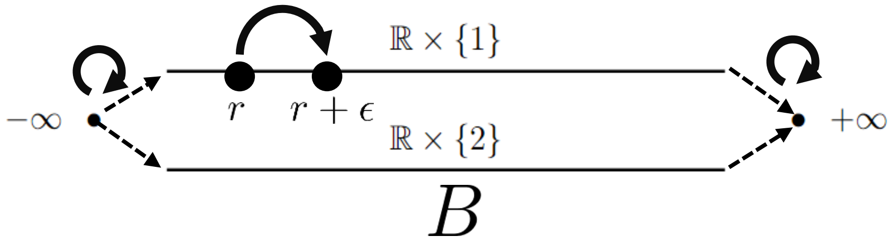


Stability theorem for bipath PD: Bottleneck distance

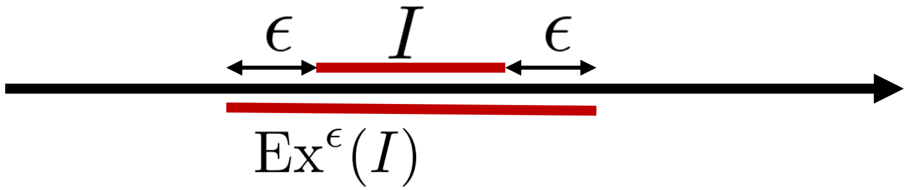
Notation

- $\Lambda := \{\Lambda_\epsilon : B \to B\}_{\epsilon \in \mathbb{R}}$: A family of order preserving maps satisfying:

$$\Lambda_\epsilon(\pm\infty) = \pm\infty, \text{ and } \Lambda_\epsilon((r,i)) = (r+\epsilon, i) \text{ for } (r,i) \in \mathbb{R} \times \{i\}.$$



- $\mathrm{Ex}^\epsilon(I) := \bigcup_{r \in [-\epsilon, \epsilon]} \Lambda_r(I)$ for an interval I.



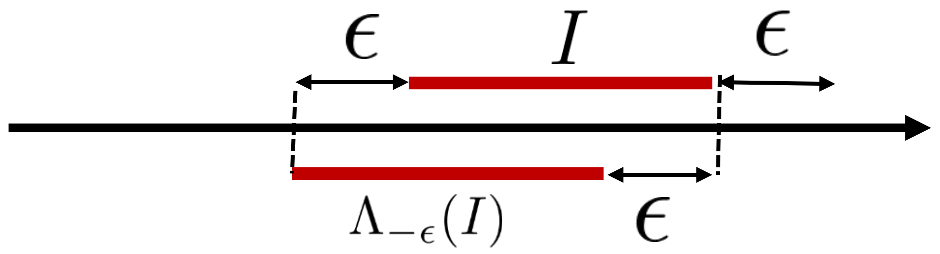
Stability theorem for bipath PD: Bottleneck distance

Proposition

For an interval I, we have $I \cap \Lambda_{-\epsilon}(I) = \emptyset$ if and only if

$$I \subseteq \mathbb{R} \times \{1\} \text{ with } I \cap \{(r-\epsilon, 1) \mid (r,1) \in I\}, \text{ or}$$
$$I \subseteq \mathbb{R} \times \{2\} \text{ with } I \cap \{(r-\epsilon, 2) \mid (r,2) \in I\}.$$

In this case, we say that I is <u>*ϵ-trivial.*</u>



if $I \subseteq \mathbb{R} \times \{i\}$ ($i = 1$ or $i = 2$)

Stability theorem for bipath PD: Bottleneck distance

Definition (ϵ-matching)

Let X and Y be multisets of intervals. We define ϵ-matching between X and Y by a partial bijection $\sigma : X \nrightarrow Y$ satisfying the following:

- Every $I \in X \setminus \mathrm{coim}\sigma$ is ϵ-trivial.
- Every $J \in Y \setminus \mathrm{im}\sigma$ is ϵ-trivial.
- If $\sigma(I) = J$, then $I \subseteq \mathrm{Ex}^{\epsilon}(J)$ and $J \subseteq \mathrm{Ex}^{\epsilon}(I)$.

Definition (Bottleneck distance)

The bottleneck distance d_{B} between a multisets of intervals in B is given by

$$d_{\mathrm{B}}(X, Y) := \inf\{\epsilon \geq 0 \mid \exists \epsilon\text{-matching between } X \text{ and } Y\}$$

for multisets of intervals X and Y in B.

Stability theorem for bipath PD: Theorem

- $f : X \to B$: A bipath function.
- $V_f := H_q(\ \cdot\ ; k) \circ (f \leq \cdot)$: Bipath PH.

Condition

For two bipath functions f, g , we set the following condition:

$$f_1^{-1}(\{-\infty\}) \overset{\text{def}}{=} f_2^{-1}(\{-\infty\}) \overset{\blacklozenge}{=} g_1^{-1}(\{-\infty\}) \overset{\text{def}}{=} g_2^{-1}(\{-\infty\}) \quad (\blacklozenge)$$

Theorem [T](Stability theorem for bipath PD).

Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be bipath functions satisfying (◆).
Then, we have the following inequality:

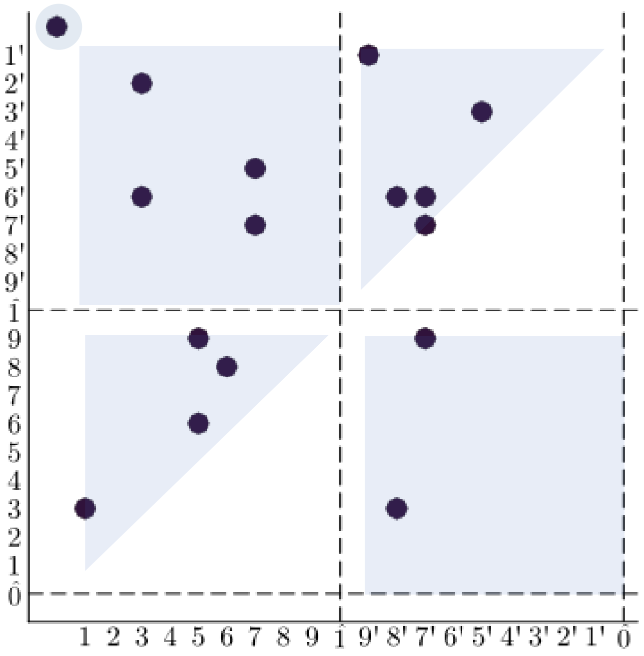
$$d_{\mathrm{B}}(\mathcal{B}(V_f), \mathcal{B}(V_g)) \leq ||f - g||_\infty.$$

- $||f - g||_\infty := \max\{||f_1 - g_1||_\infty, ||f_2 - g_2||_\infty\}.$

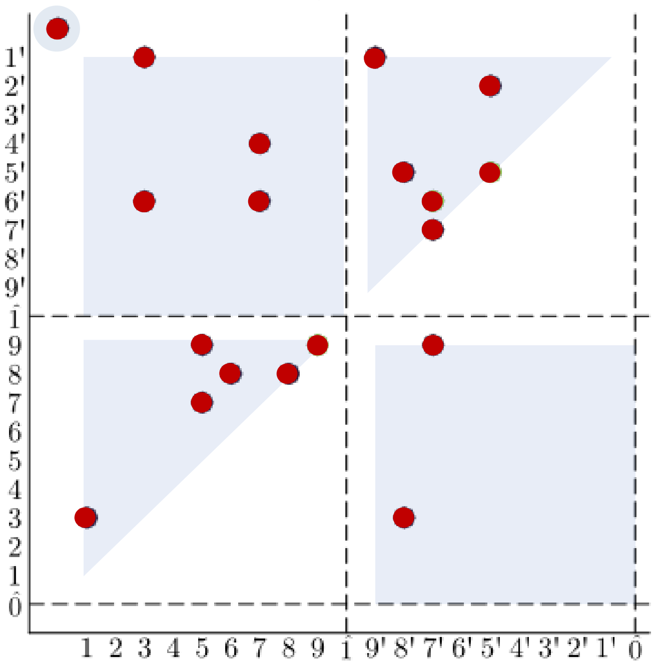
Stability theorem for bipath PD: Example using implementation

- ι: $B_{n,m} \hookrightarrow B$
- f : bipath function on a simplicial complex.
- $f' = f + \mathrm{noise}$ such that $\|f - f'\|_\infty = 1$ with (◆).

f ➡ $\mathcal{B}(V_f)$



f' ➡ $\mathcal{B}(V_{f'})$



Stability theorem for bipath PD: Example using implementation

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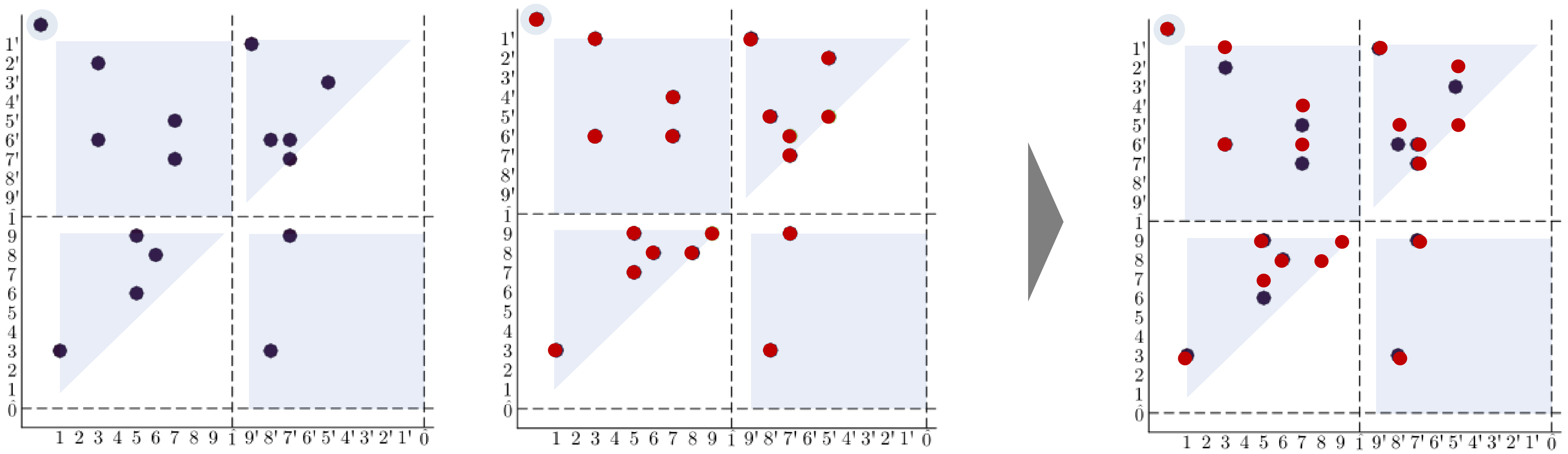

overlay PDs

Summary

We have proposed a new setting called bipath PH, which is a generalization of standard PH.

	Bipath PH
Interval-Decomposability	○
Visualization(Bipath PD)	○
Algorithm(Implementation)	○
Stability theorem for Bipath PD	○
Application	-

We are looking for ideas of applications of bipath PH!

Thank you for listening!