パーシステントホモロジーにおける区間表現のホモロジー代数的性質

Shunusuke Tada

Graduate School of Human Development and Environment
Kobe University

Joint work with

Toshitaka Aoki (Kobe), Emerson G. Escolar (Kobe)

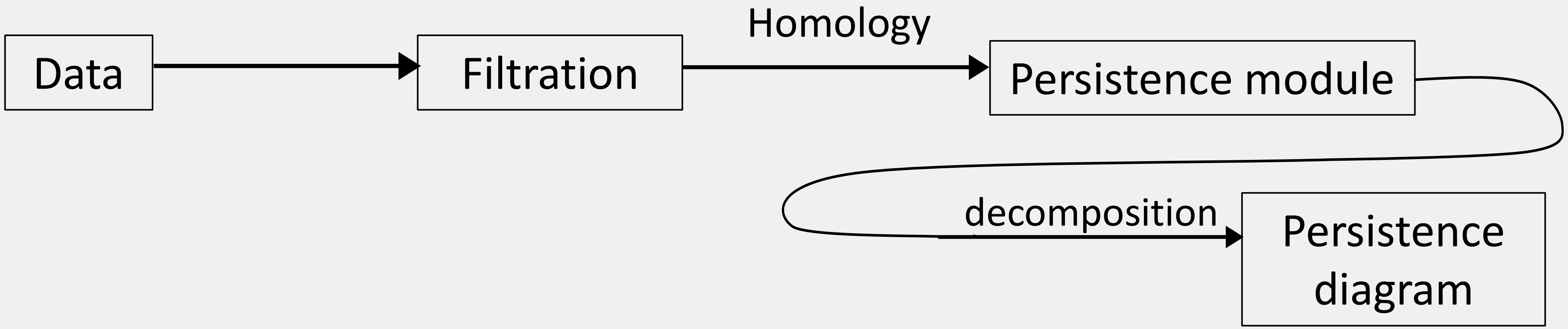
Preprint Summand-injectivity of interval approximations and monotonicity of interval global dimension. Toshitaka Aoki, Emerson G. Escolar, Shunsuke Tada. arXiv:2308.14979.

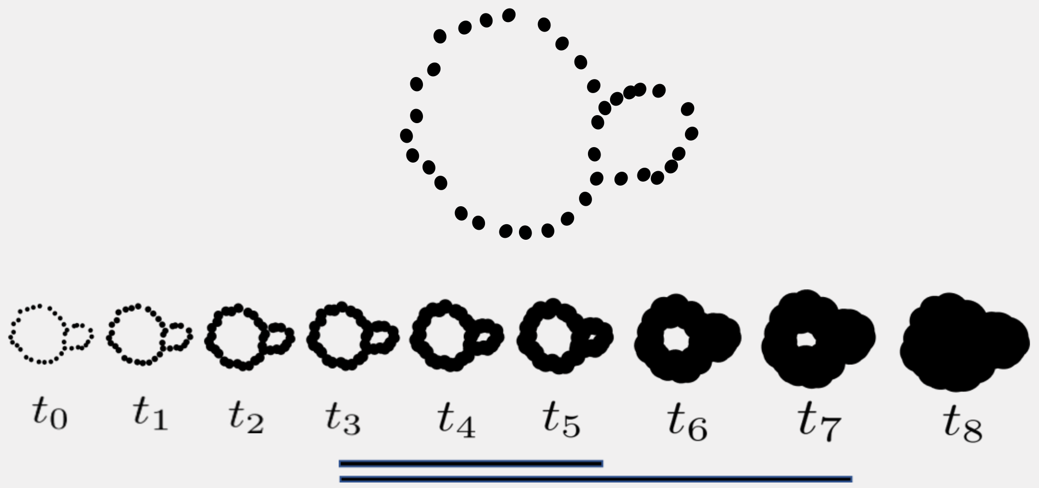
Contents

- Persistent homology
- Three results

Persistent homology

Focus on the "persistence" of the shape (connected components, holes or voids) of data.

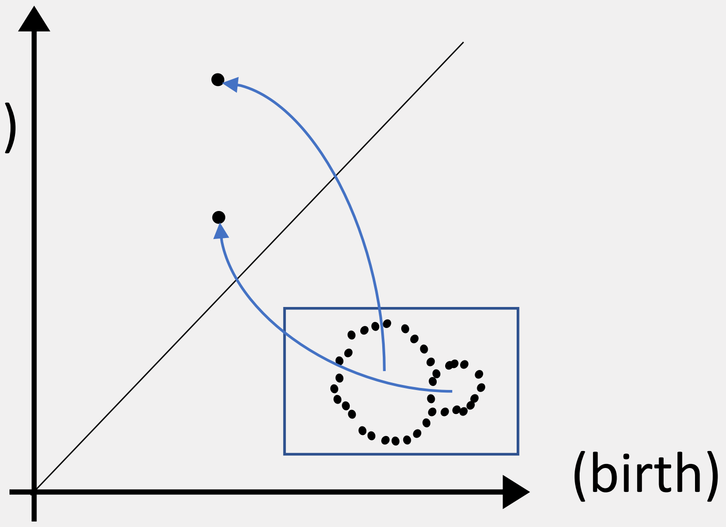




$H_1(-,k)$

Persistence module
(over A-type quiver)

$V : V_0 \to \cdots \to V_8$



Persistence diagram

Decomposition
(Corollary of Gabriel's theorem)

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i,d_i}}$$

$$I[b_i, d_i] \quad \cdots \to 0 \to \overset{b_i}{k} \to \cdots \to \overset{d_i}{k} \to 0 \to \cdots$$

Examples of applications

- Material science
- Evolutional biology
- Cosmology (cosmic web)

and others…

・Hiraoka, Y., Nakamura, T., Hirata, A., Escolar, E. G., Matsue, K., & Nishiura, Y. (2016). Hierarchical structures of amorphous solids characterized by persistent homology. *Proceedings of the National Academy of Sciences*, *113*(26), 7035-7040.

・Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan. Topology of viral evolution. Proceedings of the National Academy of Sciences, 110(46):18566–18571, 2013

・Thierry Sousbie. The persistent cosmic web and its filamentary structure–I. theory and implementation. Monthly Notices of the Royal Astronomical Society, 414(1):350–383, 201.

and others.

Zigzag persistent homology

$\bullet \to \bullet \to \bullet \to \bullet \to \bullet$

$X_1 \subset X_2 \subset X_3 \subset X_4 \subset X_5$

$\Rightarrow$

$\bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \to \bullet$

$Y_1 \supset Y_2 \subset Y_3 \supset Y_4 \subset Y_5$

(Corollary of Gabriel’s theorem)

・Carlsson, Gunnar, and Vin De Silva. “Zigzag persistence.” Foundations of computational mathematics 10 (2010): 367-405.

・Botnan, Magnus, and Michael Lesnick.“Algebraic stability of zigzag persistence modules." Algebraic & geometric topology 18.6 (2018): 3133-3204.

・McDonald, R Neuhausler, R Robinson, M Larsen, L Harrington, H Bruna, M “Zigzag persistence for coral reef resilience using a stochastic spatial model.” Journal of the Royal Society, Interface volume 20 issue 205 20230280-(23 Aug 2023).

… and others.

Question about Interval decomposability

P: Posets of type A

$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$

$\bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet$

Modules are interval decomposable (by Corollary of Gabriel's theorem)

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i,d_i}}$$

P: ?

Modules over P are interval decomposable.

Results

(1)Classification of posets
(2)Direct summand injectivity
(3)Monotonicity

Persistence module (1/2)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leqq b \Leftrightarrow \exists\,!\ a \to b$)

Persistence module (1/2)

- Let P be a finite partially ordered set (poset).
  (we see it as a category by $a \leqq b \Leftrightarrow \exists ! \; a \to b$)

- *Persistence modules over* P are functors from P to k-mod.
  (or equivalently modules over incidence algebra $k[P]$).

Example.
$V_1 \to V_2 \to V_3$ is a module over P: $1 \to 2 \to 3$,
where each V_i is a finite dimensional k-vector space.

Intervals (2/2)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

$$I \subset P : \quad \begin{tikzcd}[column sep=small, row sep=small] \bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\ & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\ & & \bullet \arrow[ur] \arrow[dr] & & \\ & & & \bullet & \end{tikzcd}$$

$$k_I : \quad \begin{tikzcd}[column sep=small, row sep=small] k \arrow[dr] & & k \arrow[dr] \arrow[dd, phantom, "\circlearrowleft"] & & 0 \\ & k \arrow[ur] \arrow[dr] & & k \arrow[ur] & \\ & & k \arrow[ur] \arrow[dr] & & \\ & & & k & \end{tikzcd}$$

Intervals (2/2)

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  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

- For an interval I of P, the *interval module* k_I is defined by
  $k_I(p) \coloneqq k$ for $p \in I$, otherwise $k_I(p) \coloneqq 0$,
  $k_I(a \to b) \coloneqq \mathrm{id}_k$ for $a, b \in I$, otherwise 0.

$$
I \subset P: \quad
\begin{tikzcd}
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd}
\qquad\qquad
k_I: \quad
\begin{tikzcd}
k \arrow[dr] & & k \arrow[dr] & & 0 \\
& k \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & k \arrow[ur] & \\
& & k \arrow[ur] \arrow[dr] & & \\
& & & k &
\end{tikzcd}
$$

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- A module is *interval decomposable* if the module decomposes into interval modules.

$$I \subset P : \quad \begin{tikzcd}[column sep=small, row sep=small]
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \circ \\
& \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \\
& & \bullet \arrow[ur] \arrow[dr] & & \\
& & & \bullet &
\end{tikzcd}
\qquad\qquad
k_I : \quad \begin{tikzcd}[column sep=small, row sep=small]
k \arrow[dr] & & k \arrow[dr] & & 0 \\
& k \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & k \arrow[ur] & \\
& & k \arrow[ur] \arrow[dr] & & \\
& & & k &
\end{tikzcd}$$

Gabriel's theorem for type A poset

For a type A poset

$$1 \leftrightarrow \cdots \leftrightarrow n,$$

and a module

$$V: V_1 \leftrightarrow \cdots \leftrightarrow V_n,$$

we have a unique decomposition of V

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i, d_i}},$$

where $I[b_i, d_i] := \cdots \leftrightarrow 0 \leftrightarrow \overset{b_i}{k} \underset{\mathrm{id}}{\leftrightarrow} \cdots \underset{\mathrm{id}}{\leftrightarrow} \overset{d_i}{k} \leftrightarrow 0 \leftrightarrow \cdots.$

Question about Interval-decomposability

P: Posets of type *A* ⇒ Modules over *P* are interval decomposable (by Corollary of Gabriel's theorem).

$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$

$\bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet$

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i, d_i}}$$

P: ? ⇐ Modules over *P* are interval decomposable.

Next Theorem classifies all the finite poset whose modules are interval-decomposable.

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

(a) Every module over P is interval-decomposable.
(b) The Hasse diagram of P is one of the following form:

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & \circlearrowleft & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

A_n (a)

$C_{n,m}$: *bipath poset* of size (n, m)

Theorem 1 [Aoki-Escolar-T]

In particular, the intervals in $C_{n,m}$ are following forms.

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\hat{0} \in I$, $\hat{1} \in I$ (all)

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\hat{0} \in I$, $\hat{1} \notin I$

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\hat{0} \notin I$, $\hat{1} \in I$

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

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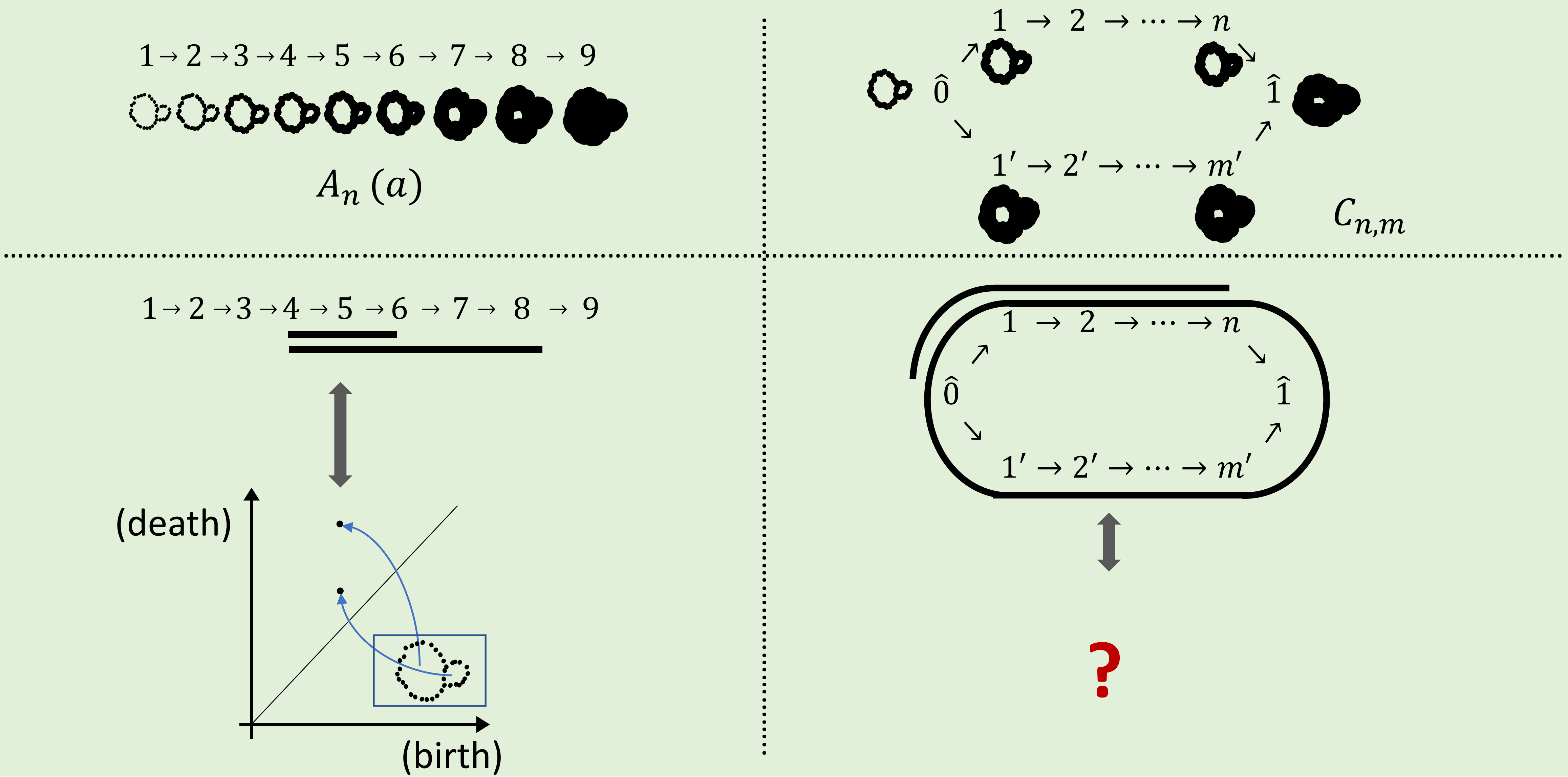
$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
\hat{0} \arrow[ur] \arrow[dr] & & & & & \hat{1} \\
 & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] &
\end{tikzcd}$$

$\hat{0} \notin I$, $\hat{1} \notin I$

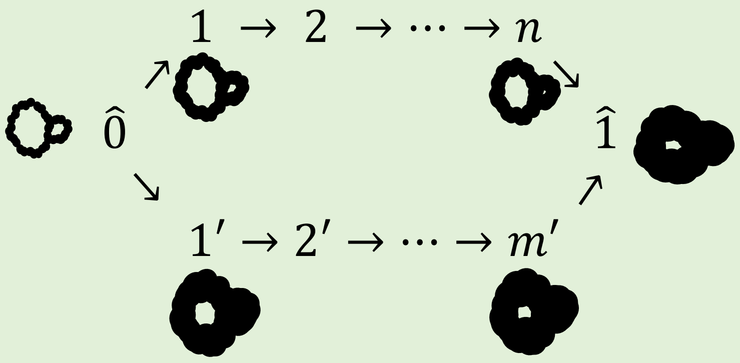
The number of intervals is

$$\frac{n^2 + 4nm + m^2 + 5n + 5m + 6}{2}.$$

Comparison between type A and $C_{n,m}$



Idea for application of $C_{n,m}$



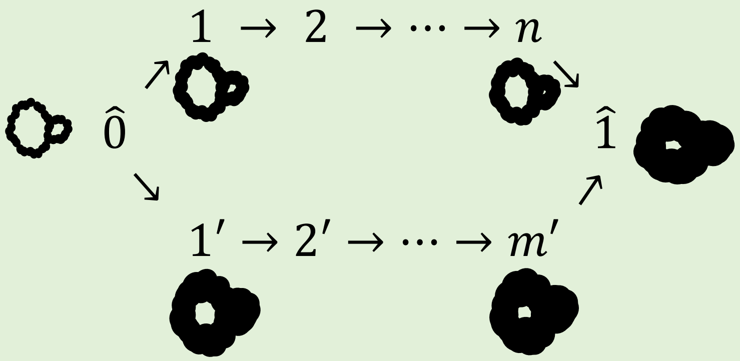
$C_{n,m}$

$$\begin{array}{cccccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to & \bullet & \to
\end{array}$$

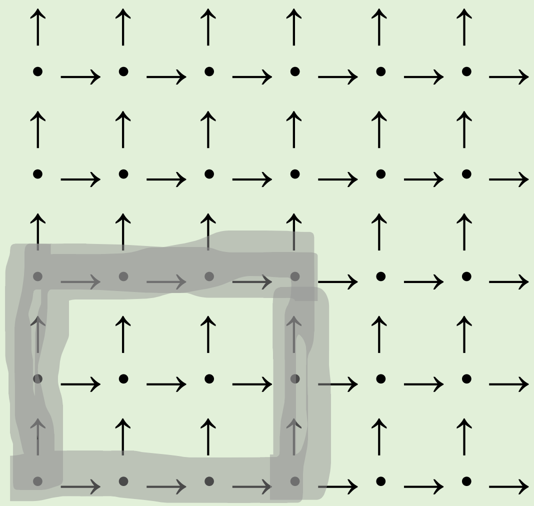
・We can get a part of information of multi-parameter persistence modules by restricting multi-filtration to $C_{n,m}$ (like fibered barcode?).

・By using two functions, we can see the robustness of shape of data in terms of the two functions .

Idea for application of $C_{n,m}$



$C_{n,m}$



$C_{6,6}$

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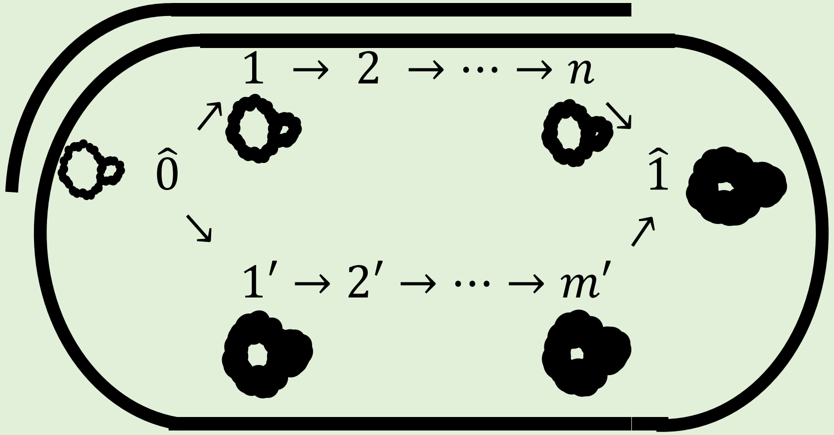
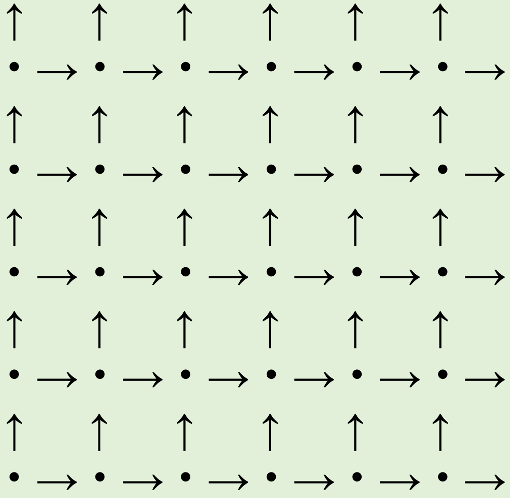
Idea for application of $C_{n,m}$



$C_{n,m}$



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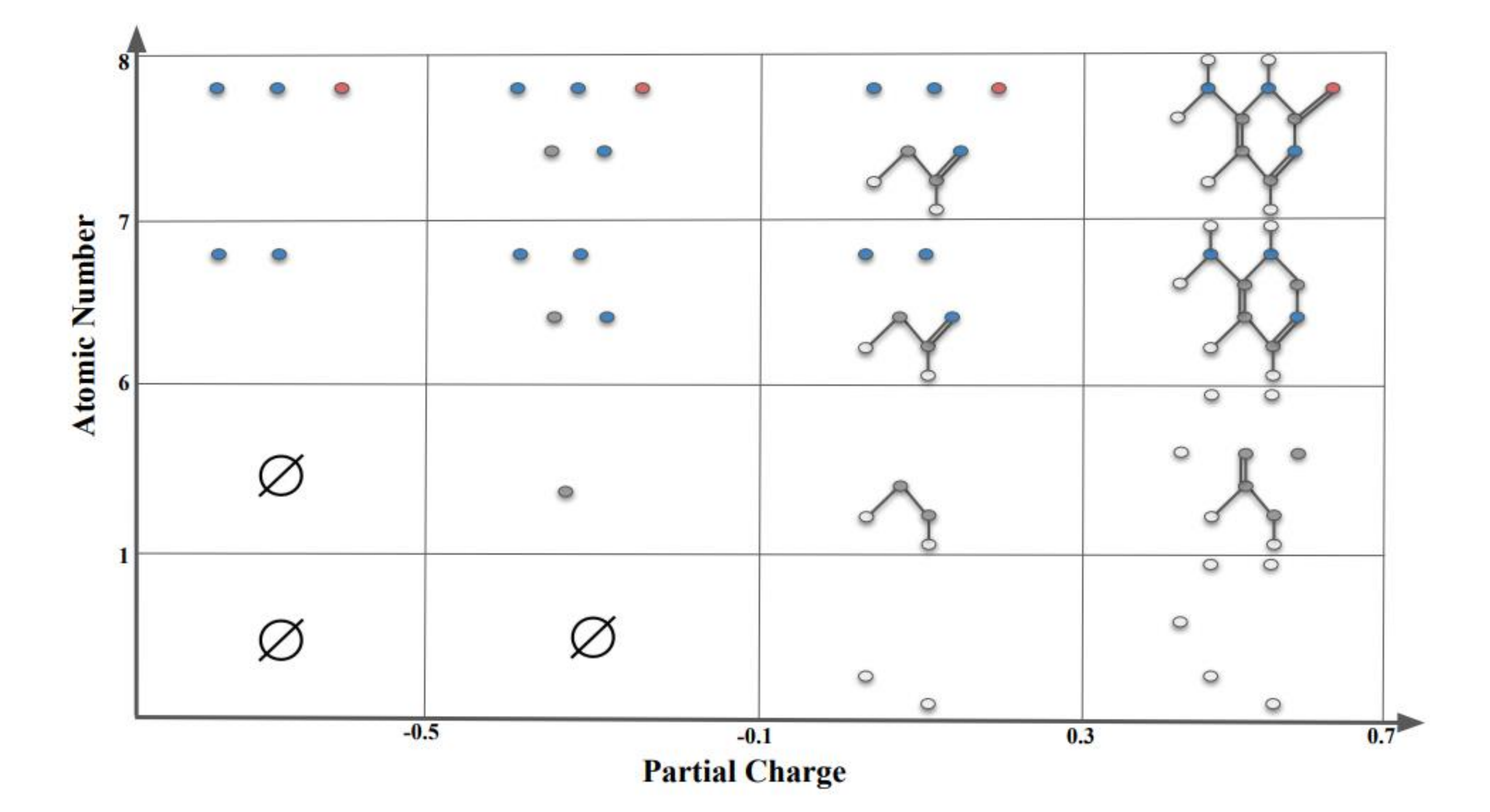
Results

(1)Classification of posets
(2)Direct summand injectivity
(3)Monotonicity

Multi-parameter persistent homology



Cytosine

Demir, Andac, et al. "ToDD: Topological compound fingerprinting in computer-aided drug discovery." *Advances in Neural Information Processing Systems* 35 (2022): 27978-27993.

Multi-parameter persistent homology

$$
\begin{array}{cccccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to & \cdot & \to
\end{array}
$$

It is difficult to classify all the indecomposable module (wild representation type)

Multi-parameter persistent homology

Indecomposable

$$
\begin{tikzcd}
K \arrow[r] & K \arrow[r] & K & & & & & & & & \\
K \arrow[r] \arrow[u] & K \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[u] & K^2 \arrow[r, "{[1\ 0]}"] \arrow[u, "{[1\ 1]}"'] & K \arrow[r] & K & & & & & & \\
K \arrow[r] \arrow[u] & K \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[u] & K^2 \arrow[r] \arrow[u] & K^2 \arrow[r] \arrow[u, "{[1\ 0]}"'] & K^2 \arrow[r, "{[0\ 1]}"] \arrow[u, "{[1\ 0]}"'] & K \arrow[r] & K & & & & \\
 & & & & K \arrow[r] \arrow[u, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"'] & 0 \arrow[r] \arrow[u] & K \arrow[u] & & & & \\
 & & & & K \arrow[r] \arrow[u] & K \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[u] & K^2 \arrow[r] \arrow[u, "{[0\ 1]}"'] & K^2 \arrow[r] & K^2 \arrow[r, "{[0\ 1]}"] & K \arrow[r] & K \\
 & & & & & & K \arrow[r] \arrow[u, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] & K \arrow[r, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \arrow[u, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] & K^2 \arrow[r, "{[0\ 1]}"] \arrow[u] & K \arrow[r] \arrow[u] & K \arrow[u] \\
 & & & & & & & & K \arrow[r] \arrow[u, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"'] & K \arrow[r] \arrow[u] & K \arrow[u]
\end{tikzcd}
$$

M.Buchet, Emerson G. Escolar “Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module” *Journal of Applied and Computational Topology*

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n\ (a)$

$$\begin{array}{ccccc} & \nearrow & 1 \to 2 \to \cdots \to n & \searrow & \\ \hat{0} & & \circlearrowleft & & \hat{1} \\ & \searrow & 1' \to 2' \to \cdots \to m' & \nearrow & \end{array}$$

$C_{n,m}$

(Modules are always interval decomposable)

We want to understand complex modules (obtained from data) over posets, using “good” modules (interval modules).

Interval approximation

・Benjamin Blanchette, Thomas Brüstle, and Eric J Hanson. “Homological approximations in persistence theory.” *Canadian Journal of Mathematics*, pages 1–38, 2021.

・Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. “Approximation by interval-decomposables and interval resolutions of persistence modules.” *Journal of Pure and Applied Algebra*, 227(10):107397, 2023.

Interval approximation (1/1)

- $\mathcal{J}_P$ is the set of interval decomposable modules over P.
- M is a module over P.

An *interval approximation of M* is a morphism $f: J \to M$ with $J \in \mathcal{J}_P$ s. t. for any $g: I \to M$ with $I \in \mathcal{J}_P$ factor through f.

$$
\begin{tikzcd}
\mathcal{J}_P \ni I \arrow[d, dashed, "\exists"'] \arrow[dr, "\forall g"] & \\
\mathcal{J}_P \ni J \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & M
\end{tikzcd}
$$

An *interval cover* of M is an interval approximation such that the number of direct summands of the domain is smaller than that of other interval approximations (uniquely determined).

Question

How can we calculate an interval cover of any modules (easily)?

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<u>Calculation of an interval cover of a module *M*</u>

(1) We have an interval approximation $\bigoplus_{I \in \mathbb{I}(P)} k_I^{m_I} \to M$, where m_I is k-dimension of $\mathrm{Hom}_k(k_I, M)$ and $\mathbb{I}(P)$ is the set of all the intervals in the poset *P*.

(2) We reduce the direct summands of the above interval approximation until we obtain the interval cover.

We give a helpful observation to the Question.

Theorem 2 [Aoki-Escolar-T]

Let P be a finite poset and M be a module over P. For any interval cover of M

$$f = (f_i) : \bigoplus_{i=1}^{n} k_{I_i} \to M,$$

the following holds.

(1) f is surjective.
(2) Each $f_i : k_{I_i} \to M$ is injective.
(3) For every $a \in P$, we have $M(a) = 0$ if and only if $\left(\bigoplus_{i=1}^{n} k_{I_i}\right)(a) = 0$.

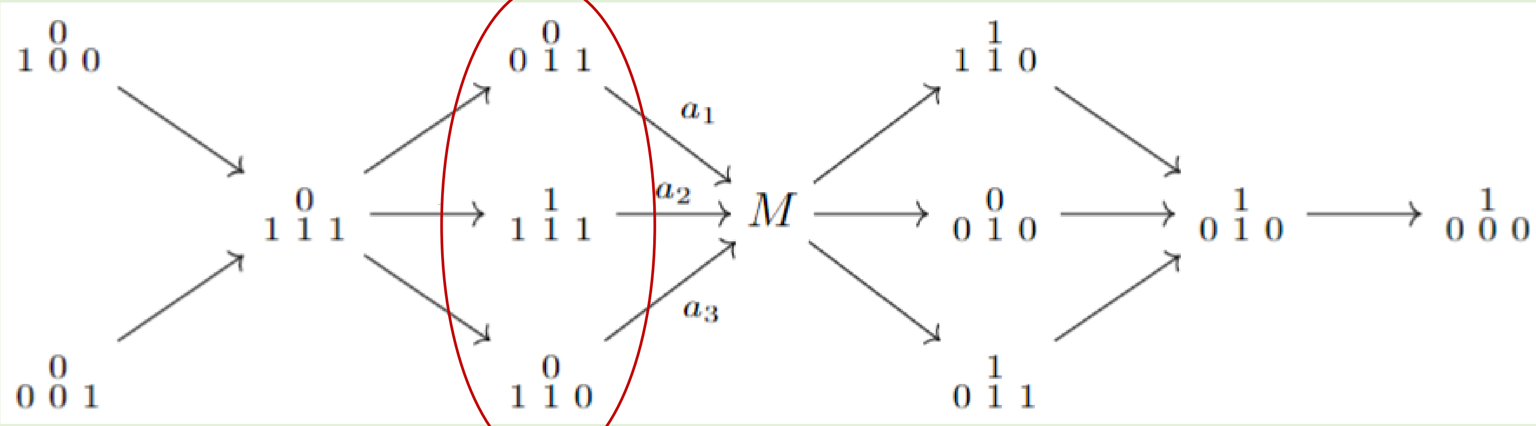
In particular, every k_{I_i} is given by an interval submodule of M.

Remark

Recently, [Asashiba, 2023, Proposition 4.8, arXiv:2307.06559] gave the essentially same result (see also [Blanchette-Brüstle-Hanson, Proposition 6.7, 2021, Canadian Journal of Mathematics, 1-38]).

Example of interval cover and injectivity

$$P := \quad \begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \leftarrow & \bullet & \rightarrow & \bullet \end{array}, \qquad M := \quad \begin{array}{ccccc} & & k & & \\ & & \Big\downarrow {\scriptstyle \begin{bmatrix}1\\1\end{bmatrix}} & & \\ k & \xleftarrow[{[1,0]}]{} & k^2 & \xrightarrow[{[0,1]}]{} & k. \end{array}$$



An interval cover of M is given by

$$\begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \end{array} \oplus \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \end{array} \oplus \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \end{array} \xrightarrow{[a_1, a_2, a_3]} M$$

Results

(1) Classification of posets

(2) Direct summand injectivity

(3) Monotonicity

Resolution dimension (1/2)

- An *interval resolution of M* is an exact sequence

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & J_m \arrow[r, "g_m"] & \cdots \arrow[r] & J_2 \arrow[rr, "g_2"] \arrow[dr, dashed, "f_2"] & & J_1 \arrow[rr, "g_1"] \arrow[dr, dashed, "f_1"] & & J \arrow[r, "f"] & M \arrow[r] & 0, \\
& & \cdots\ K_3 \arrow[ur, dashed, "\iota_3"] & & K_2 \arrow[ur, dashed, "\iota_2"] & & K_1 \arrow[ur, dashed, "\iota_1"] & & &
\end{tikzcd}
$$

then we say that the *interval resolution dimension of M* is m and write int-res-dim M = m.

*int-res-dim M = 0 $\Longleftrightarrow$ M is interval-decomposable.

Interval resolution global dimension (2/2)

- *interval resolution global dimension of P* is

int-res-gldim(P) :=sup{int-res-dim$(M)\,|\,M$: modules over P}

* [Asashiba-Escolar-Nakashima-Yoshiwaki, 23] show that int-res-gldim$(P) < \infty$ for any finite poset P.

*int-res-gldim(P) = 0 $\Longleftrightarrow$ Any module over P is interval-decomposable.

Remark

Interval resolution global dimension (2/2)

- *interval resolution global dimension of P* is

int-res-gldim(P) $:=$sup{int-res-dim(M) | M: modules over P}

* [Asashiba-Escolar-Nakashima-Yoshiwaki, 23] show that int-res-gldim(P) $< \infty$ for any finite poset P.

*int-res-gldim(P) = 0 $\Leftrightarrow$ Any module over P is interval-decomposable.

Remark

int-res-gldim(P) is zero if and only if the Hasse diagram of P is either (i) or (ii),

(i) $1 \longleftrightarrow \cdots \longleftrightarrow n$, (ii) $\begin{tikzcd} & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ \hat{0} \arrow[ur] \arrow[dr] & & & \circlearrowleft & & \hat{1} \\ & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd}$.

We can say Theorem 1 gives a complete classification of finite posets with interval resolution global dimension zero.

Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P).$$

Remark

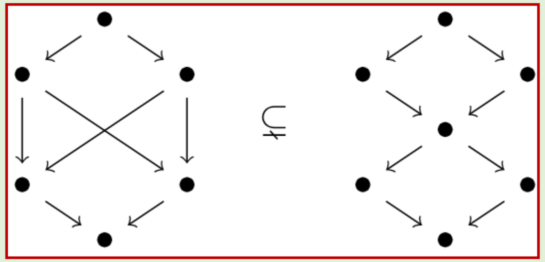
Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P).$$

Remark

The above monotonicity does not hold for (usual) global dimension in general [Igusa-Zacharia, 1990].



Poset	Q	$\subset$	P	
Global dimension	3	$>$	2	
int-res-gldim	1	$<$	2	(over a field with two elements)

Sketch of proof

For a full subposet Q of P, we have an isomorphism

$$k[Q] \cong ek[P]e$$

of k-algebras, where $e \coloneqq \Sigma_{x \in Q} e_x$. It induces adjoint functors

$$\begin{tikzcd}
\operatorname{mod} k[P] \arrow[rr, "\mathrm{Res}"] & & \operatorname{mod} k[Q] \arrow[ll, bend right, "\mathrm{T} \coloneqq - \otimes_{k[Q]} ek[P]"'] \arrow[ll, bend left, "\mathrm{L} \coloneqq \mathrm{Hom}_{k[Q]}(ek[P], -)"]
\end{tikzcd}$$

$$\mathrm{T} \dashv \mathrm{Res} \dashv \mathrm{L}$$

- Res preserves interval decomposability of modules.
- T and L do NOT preserve interval decomposability of modules in general.

We find a functor Θ that sends to interval modules over Q to interval modules over P by using T and L.

Sketch of proof (The functor Θ)

Using adjoint functors, we have

$$\mathrm{Hom}_{k[Q]}(M, M) \;\cong\; \mathrm{Hom}_{k[P]}\big(\mathrm{T}(M), \mathrm{L}(M)\big).$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1_M & \longmapsto & \theta_M \end{array}$$

For a given module $M \in \mathrm{mod}\, k[Q]$, let

$$\Theta(M) \coloneqq \mathrm{Im}(\theta_M).$$

It gives rise to a functor Θ. It is called *intermediate extension* in [Kuhn, 94], and *prolongedment intermédiare* in [Beilison-Bernstein-Deligne, 82].

Proposition For a given interval I of Q, let k_I be the corresponding interval $k[Q]$-module. Then, we have

$$\Theta(k_I) \cong k_{\mathrm{conv}(I)},$$

where $\mathrm{conv}(I)$ is the smallest interval of P containing I.

Sketch of proof (The functor Θ)

We obtain a pair of functors

$$\begin{tikzcd} \mathrm{mod}\, k[P] \arrow[r, bend right, "\mathrm{Res}"'] & \mathrm{mod}\, k[Q] \arrow[l, bend right, "\Theta"'] \end{tikzcd}$$

satisfying the following properties:

(i) Res preserves interval decomposability of modules.

(ii) Θ sends interval modules to interval modules by Proposition.

(iii) $1_{\mathrm{mod}\, k[Q]} \cong \mathrm{Res} \circ \Theta$.

Proposition For any $M \in \mathrm{mod}\, k[Q]$, we have the following inequality

$$\text{int-res-dim}(M) \leq \text{int-res-dim}\big(\Theta(M)\big).$$

Since M is an arbitrary module, we obtain the desired inequality

$$\text{int-res-gldim}(k[Q]) \leq \text{int-res-gldim}(k[P]).$$

□

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

(a) Every module over P is interval decomposable ($\Longleftrightarrow$ int-res-gldim(P)=0).

(b) The Hasse diagram of P is $1 \longleftrightarrow \cdots \longleftrightarrow n$ or

$$\begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ \hat{0} \arrow[ur] \arrow[dr] & & \arrow[phantom, "\circlearrowleft"] & & & \hat{1} \\ & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd}$$

$A_n\ (a)$ $\qquad$ $C_{n,m}$.

Sketch of proof (a ⇒ b)

・The Hasse diagram of P does not have a vertex with more than degree 3 (by Theorem 3).

$$\begin{tikzcd}[row sep=small, column sep=small] & \bullet \arrow[d] & \\ \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd} : Q \subseteq P \qquad \begin{tikzcd} & k \arrow[d, "{\begin{bmatrix}1\\1\end{bmatrix}}"] & \\ k & k^2 \arrow[l, "{[1,0]}"'] \arrow[r, "{[0,1]}"] & k \end{tikzcd}$$

・P is either A_n or $\tilde{A}_m$ for some n and m.

・P must be A_n or $C_{n,m}$ for some n and m.

※We prove the converse (interval decomposability of module over $C_{n,m}$) by using theory of special biserial algebra.

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

(a) Every module over P is interval decomposable ($\Leftrightarrow$ int-res-gldim(P)=0).

(b) The Hasse diagram of P is $\underbrace{1 \longleftrightarrow \cdots \longleftrightarrow n}_{A_n\ (a)}$ or

$$\begin{array}{ccccc} & & 1 \to 2 \to \cdots \to n & & \\ & \nearrow & & \searrow & \\ \hat{0} & & \circlearrowleft & & \hat{1} \\ & \searrow & & \nearrow & \\ & & 1' \to 2' \to \cdots \to m' & & \end{array} \quad C_{n,m}\,.$$

Sketch of proof (a ⇒ b)

- The Hasse diagram of P does not have a vertex with more than degree 3 (by Theorem 3).
- P is either A_n or $\tilde{A}_m$ for some n and m.
- P must be A_n or $C_{n,m}$ for some n and m.

$$\begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \leftarrow & \color{red}{\bullet} & \rightarrow & \bullet \end{array} \quad : Q \subseteq P \qquad \begin{array}{ccccc} & & k & & \\ & & \downarrow \begin{bmatrix}1\\1\end{bmatrix} & & \\ k & \xleftarrow{[1,0]} & k^2 & \xrightarrow{[0,1]} & k \end{array}$$

contradiction

1≤ int-res-gldim(Q) ≤ int-res-gldim(P) $= 0$.

※We prove the converse (interval decomposability of module over $C_{n,m}$) by using theory of special biserial algebra.

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

(a) Every module over P is interval decomposable ($\Longleftrightarrow$ int-res-gldim(P)=0).

(b) The Hasse diagram of P is $\underset{A_n\ (a)}{1 \longleftrightarrow \cdots \longleftrightarrow n}$ or

$$\begin{tikzcd}[column sep=small] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\ \hat{0} \arrow[ur] \arrow[dr] & & & \circlearrowleft & & \hat{1} \\ & 1' \arrow[r] & 2' \arrow[r] & \cdots \arrow[r] & m' \arrow[ur] & \end{tikzcd} \quad C_{n,m}\ .$$

Sketch of proof (a ⇒ b)

- The Hasse diagram of P does not have a vertex with more than degree 3 (by Theorem 3).

$$1 \longleftrightarrow \cdots \longleftrightarrow n \ : A_n$$

$$\begin{tikzcd}[column sep=small] & m+1 \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ 1 \arrow[r, leftrightarrow] & \cdots & m \arrow[l, leftrightarrow] \end{tikzcd} \ : \tilde{A}_m$$

- P is either A_n or $\tilde{A}_m$ for some n and m.
- P must be A_n or $C_{n,m}$ for some n and m.

※We prove the converse (interval decomposability of module over $C_{n,m}$) by using theory of special biserial algebra.

Summary

(1) We classified finite posets whose modules are always interval decomposable. In other words, we classified finite posets with int-res-gldim is zero.

(2) We show that restriction of each direct summand of interval cover is injective.
(It makes calculation of interval cover easier.)

(3) We show the monotonicity of int-res-gldim.
(This is used to show the first result.)

Discussion

- When $C_{n,m}$ is useful in TDA? Stability?
- Does int-res-gldim depend on the characteristic of fields?
- When do we have int-res-gldim(Q) = int-res-gldim(P) for $Q \subset P$?
- Can we calculate interval cover easily?

- Computation using GAP package QPA("Quiver and Path Algebras") and "pmgap" by E. G. Escolar to calculate modules over poset.
- Computation using Julia.

Thank you for your attention!

Our paper

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