Posets whose persistence modules are always interval decomposable and homological invariants

Shunusuke Tada

Graduate School of Human Development and Environment
Kobe University

Joint work with

Toshitaka Aoki (Kobe), Emerson G. Escolar (Kobe)

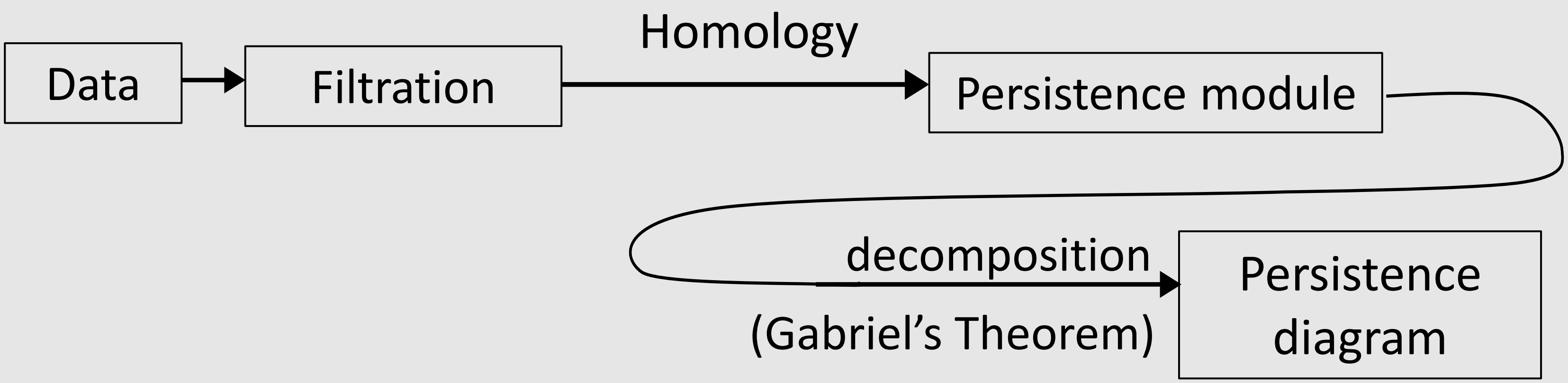
Preprint Summand-injectivity of interval approximations and monotonicity of interval global dimension. Toshitaka Aoki, Emerson G. Escolar, Shunsuke Tada. arXiv:2308.14979.

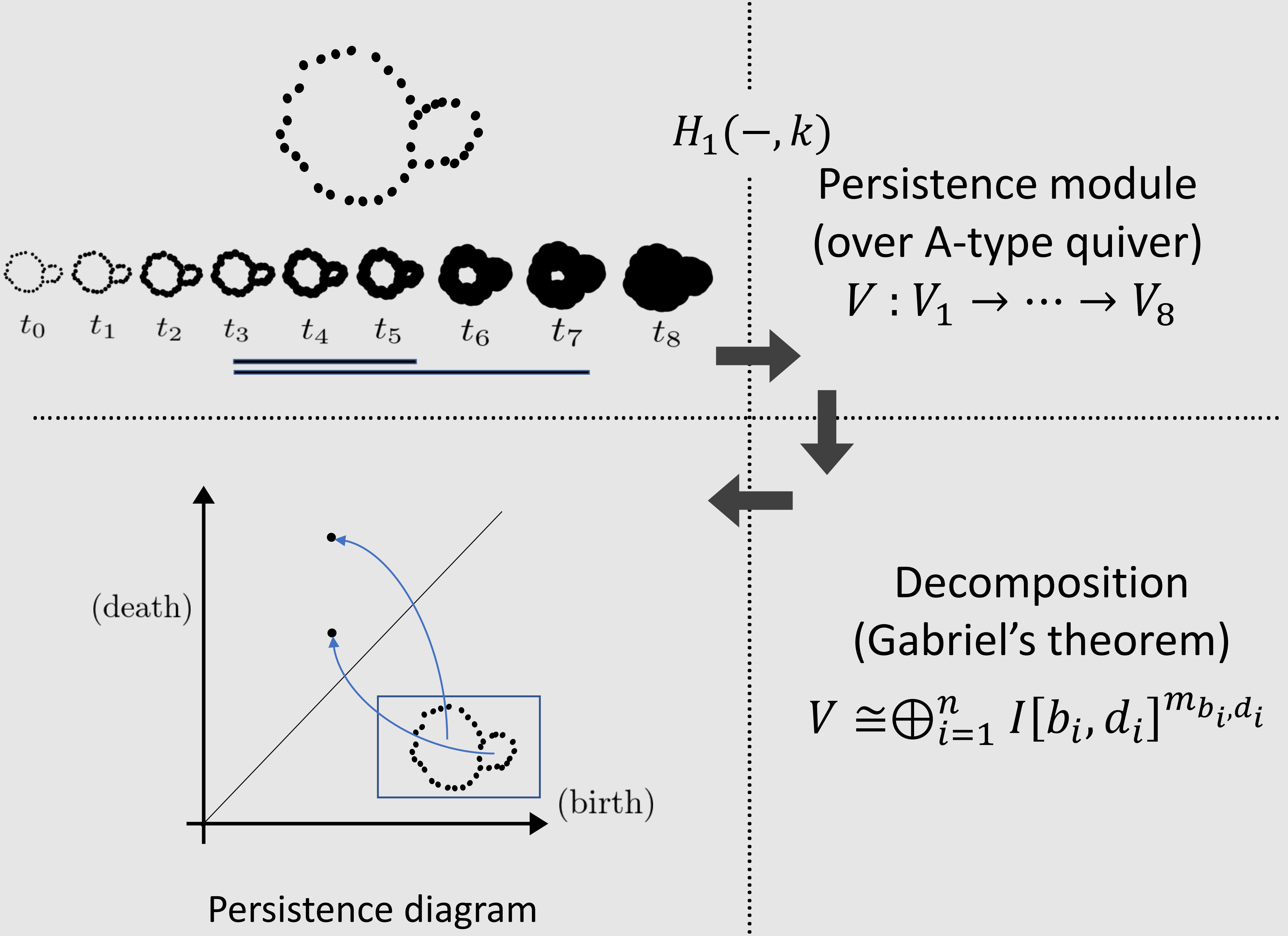
Contents

- Persistent homology？
- Three results

Persistent homology

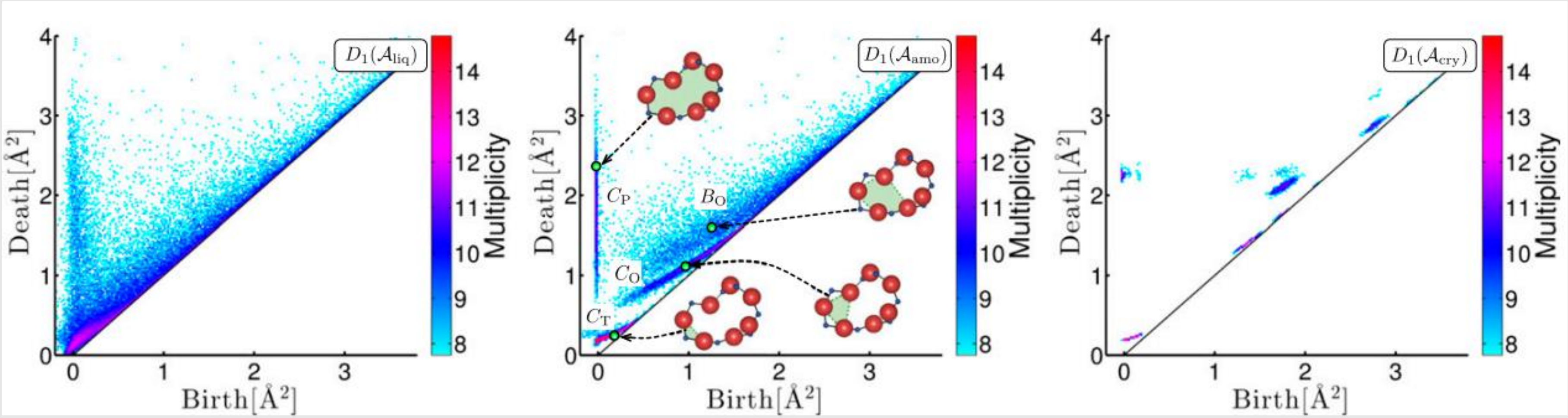
Focus on the "persistence" of the shape
(connected components, holes or voids) of data.





t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8
$H_1(-,k)$
Persistence module
(over A-type quiver)
$V : V_1 \to \cdots \to V_8$
Decomposition
(Gabriel's theorem)
$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i,d_i}}$
(death)
(birth)
Persistence diagram

Example in material science



liquid glass | amorphous glass | crystalline glass

Hiraoka, Y., Nakamura, T., Hirata, A., Escolar, E. G., Matsue, K., & Nishiura, Y. (2016). Hierarchical structures of amorphous solids characterized by persistent homology. *Proceedings of the National Academy of Sciences*, *113*(26), 7035-7040.

Zigzag persistence module / Interval decomposability

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$$
$$X_1 \subset X_2 \subset X_3 \subset X_4 \subset X_5$$

$$\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet$$
$$Y_1 \supset Y_2 \subset Y_3 \supset Y_4 \subset Y_5$$

(Gabriel's theorem works)

・Carlsson, Gunnar, and Vin De Silva. "Zigzag persistence." Foundations of computational mathematics 10 (2010): 367-405.

・Botnan, Magnus, and Michael Lesnick."Algebraic stability of zigzag persistence modules." Algebraic & geometric topology 18.6 (2018): 3133-3204.

・McDonald, R Neuhausler, R Robinson, M Larsen, L Harrington, H Bruna, M "Zigzag persistence for coral reef resilience using a stochastic spatial model." Journal of the Royal Society, Interface volume 20 issue 205 20230280-(23 Aug 2023).

… and others.

Question

P: Posets of type A ⇒ Modules over P are interval decomposable (by Gabriel's theorem).

$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$

$\bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet$

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i,d_i}}$$

P: ? ⇐ Modules over P are interval decomposable.

Results

(1)Classification of posets

(2)Direct summand injectivity

(3)Monotonicity

Persistence module (1/2)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leq b \Leftrightarrow \exists !\ a \rightarrow b$)

-

Persistence module (1/2)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leq b \Leftrightarrow \exists !\ a \rightarrow b$)

- *Persistence modules over P* are functors from P to k-mod.

  (We call them modules for short.)

Persistence module (1/2)

- Let P be a finite partially ordered set (poset).

  (we see it as a category by $a \leq b \Leftrightarrow \exists!\ a \rightarrow b$)

- *Persistence modules over P* are functors from P to k-mod.

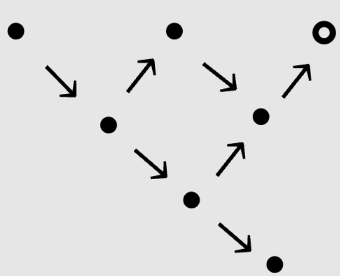
  (We call them modules for short.)

Example.
$V_1 \rightarrow V_2 \rightarrow V_3$ is a module over $P: 1 \rightarrow 2 \rightarrow 3$, where each V_i is a finite dimensional k-vector space.

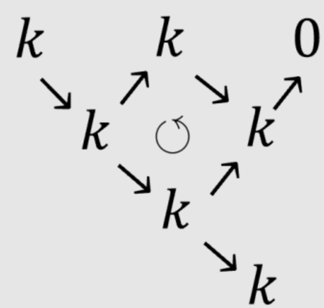
Intervals (2/2)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

$I \subset P$:



k_I :



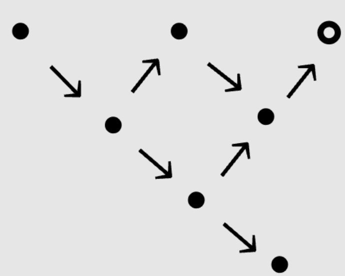
Intervals (2/2)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \le y \le z$, and $x, z \in I$ imply $y \in I$).

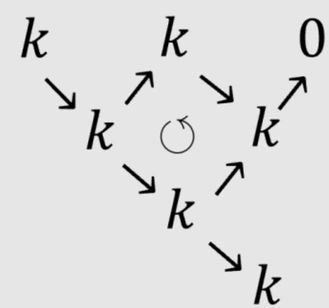
- For an interval I of P, the *interval module k_I* is defined by
  $k_I(p) \coloneqq k$ for $p \in I$, otherwise $k_I(p) \coloneqq 0$,
  $k_I(a \to b) \coloneqq \mathrm{id}_k$ for $a, b \in I$, otherwise 0.

-

$I \subset P$:



k_I :



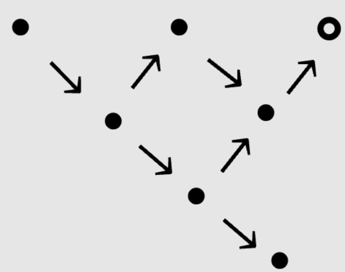
Intervals (2/2)

- A full subposet I of P is called *interval* if I is
  (1) connected (the Hasse diagram of I is connected),
  (2) convex ($x \leq y \leq z$, and $x, z \in I$ imply $y \in I$).

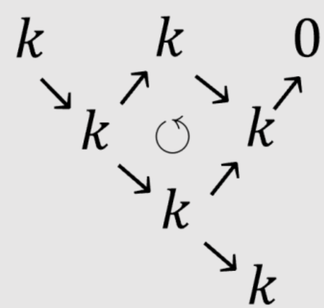
- For an interval I of P, the *interval module k_I* is defined by
  $k_I(p) \coloneqq k$ for $p \in I$, otherwise $k_I(p) \coloneqq 0$,
  $k_I(a \to b) \coloneqq \mathrm{id}_k$ for $a, b \in I$, otherwise 0.

- A module is *interval decomposable* if the module decomposes into interval modules.

$I \subset P$:



k_I :



Question

P: Posets of type *A* ⟹ Modules over *P* are interval decomposable (by Gabriel's theorem).

$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$

$\bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet \leftarrow\rightarrow \bullet$

$$V \cong \bigoplus_{i=1}^{n} I[b_i, d_i]^{m_{b_i,d_i}}$$

P: ? ⟸ Modules over *P* are interval decomposable.

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

(a) Every module over P is interval decomposable.
(b) The Hasse diagram of P is one of the following form:

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

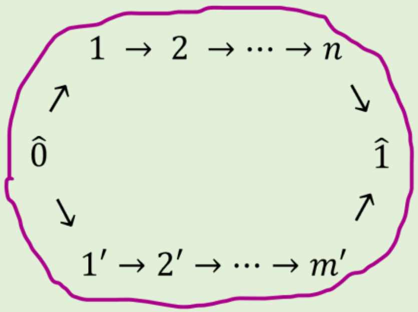
$$\begin{array}{ccccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & \circlearrowleft & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

$A_n\ (a)$

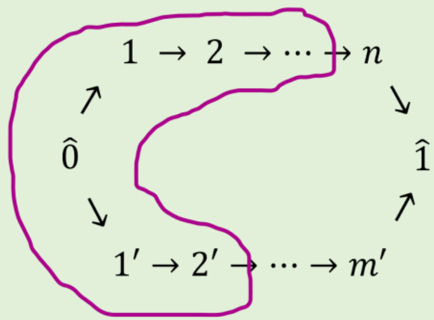
$C_{n,m}$: *bipath poset* of size (n, m)
(~~commutative cycle~~)

Theorem 1 [Aoki-Escolar-T]

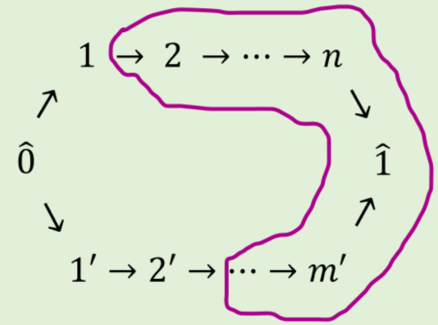
In particular, the intervals in $C_{n,m}$ are following forms.



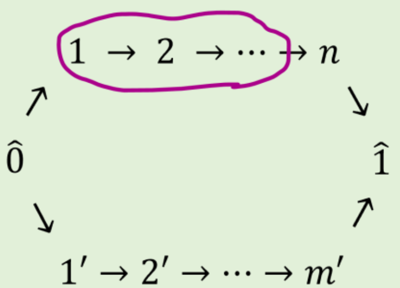
$\hat{0} \in I$, $\hat{1} \in I$ (all)



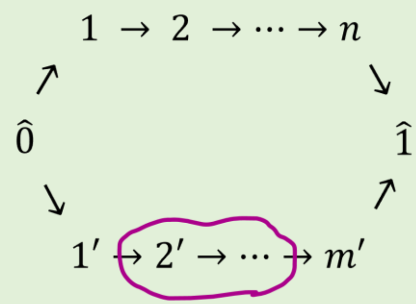
$\hat{0} \in I$, $\hat{1} \notin I$



$\hat{0} \notin I$, $\hat{1} \in I$



$\hat{0} \notin I$, $\hat{1} \notin I$



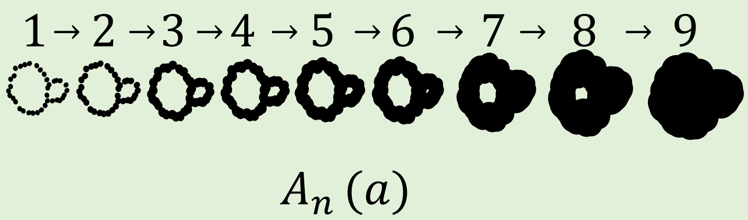
$\hat{0} \notin I$, $\hat{1} \notin I$

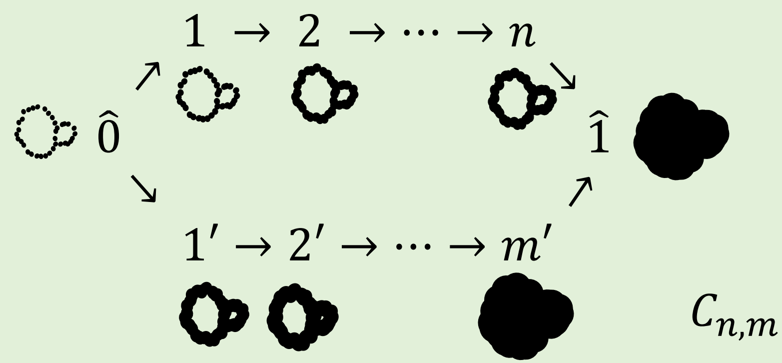
The number of intervals is

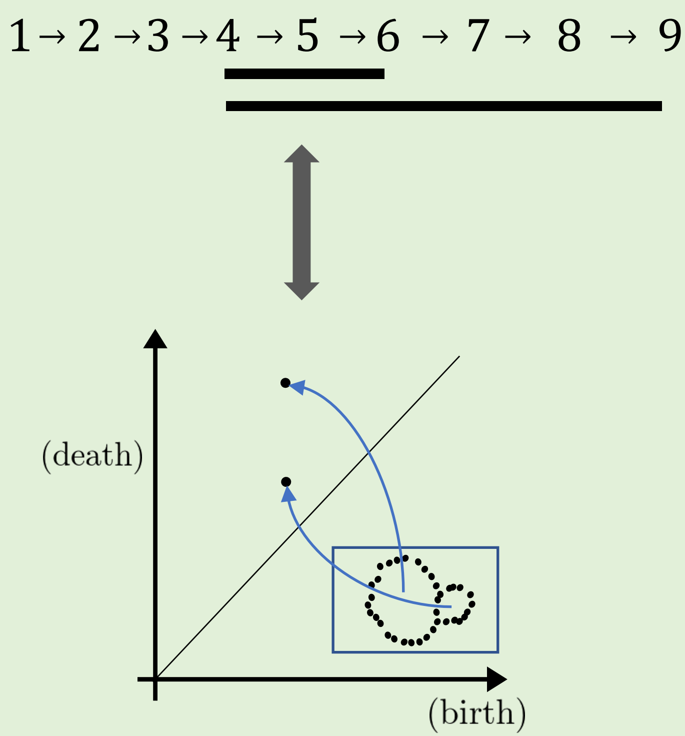
$$\frac{n^2 + 4nm + m^2 + 5n + 5m + 6}{2}.$$

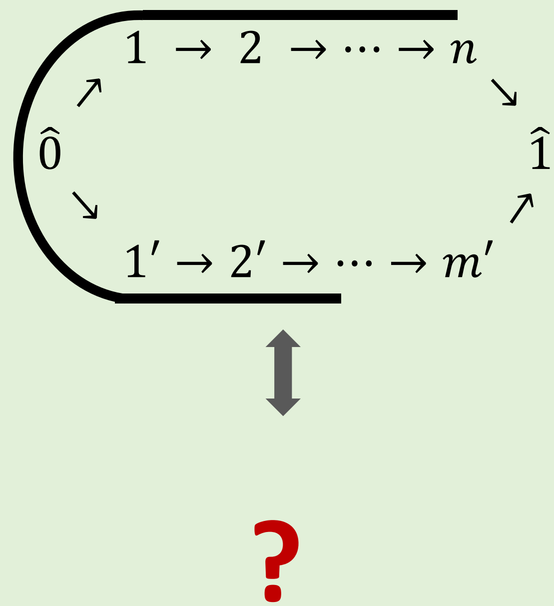
* We give a sketch of proof of Theorem 1 after in Theorem 3.

Comparison between type A and $C_{n,m}$

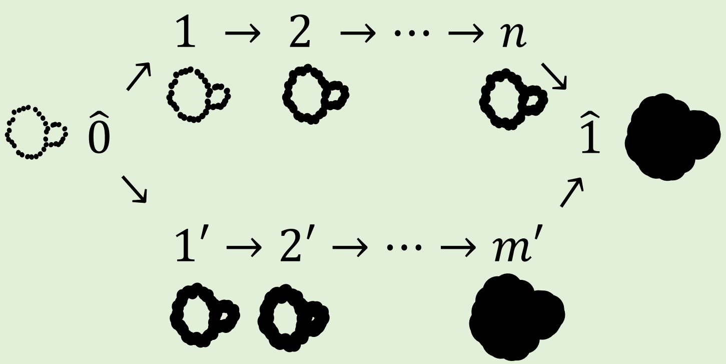




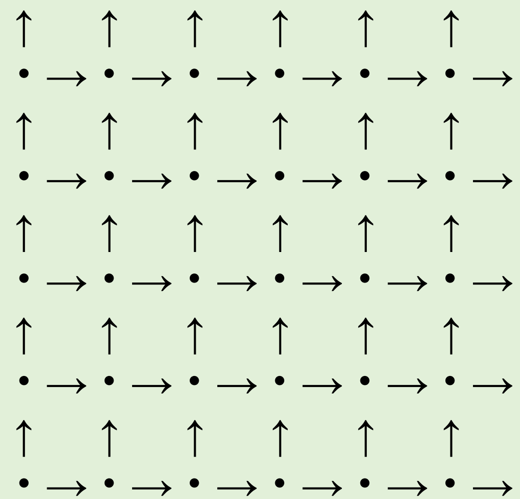




Idea for application of $C_{n,m}$



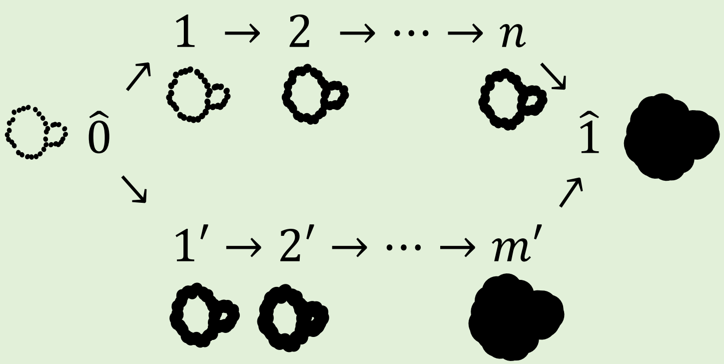
$C_{n,m}$



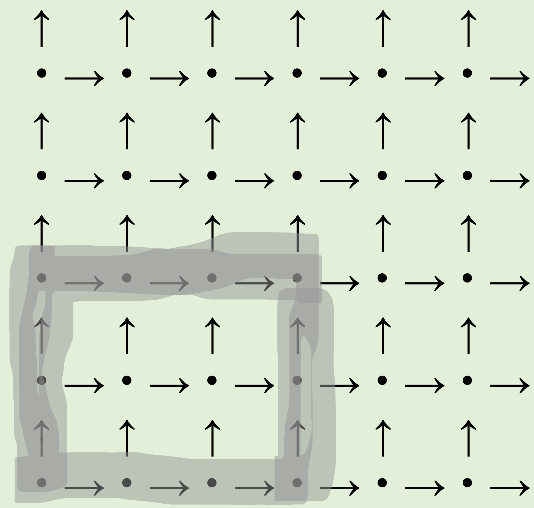
・We can get a part of information of multi-parameter persistence modules by restricting multi-filtration to $C_{n,m}$ (like fibered barcode?).

・By using two functions, we can see the robustness of shape of data in terms of the two functions .

Idea for application of $C_{n,m}$



$C_{n,m}$

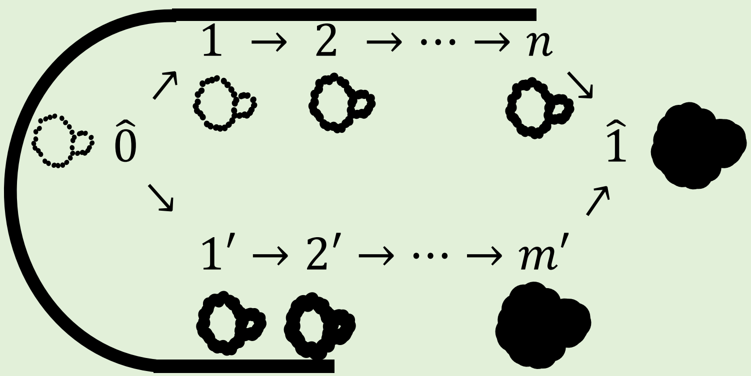


$C_{6,6}$

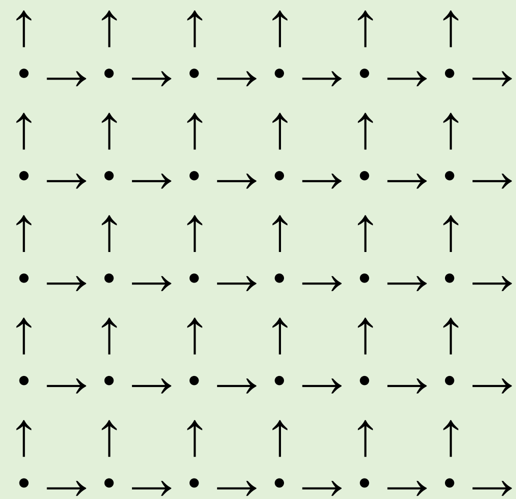
・We can get a part of information of multi-parameter persistence modules by restricting multi-filtration to $C_{n,m}$ (like fibered barcode?).

・By using two functions, we can see the robustness of shape of data in terms of the two functions .

Idea for application of $C_{n,m}$



$C_{n,m}$



$C_{6,6}$

・We can get a part of information of multi-parameter persistence modules by restricting multi-filtration to $C_{n,m}$ (like fibered barcode?).

・By using two functions, we can see the robustness of shape of data in terms of the two functions .

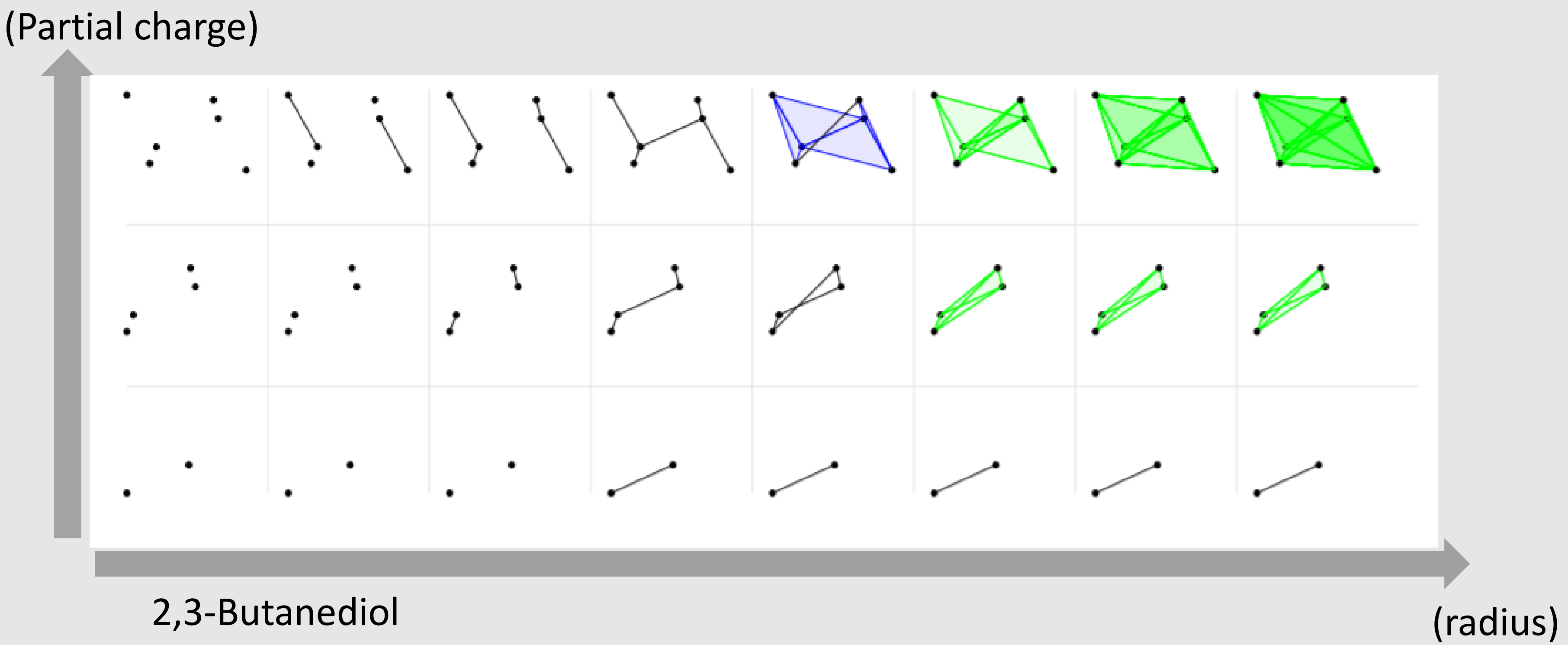
Results

(1) Classification of posets

(2) Direct summand injectivity

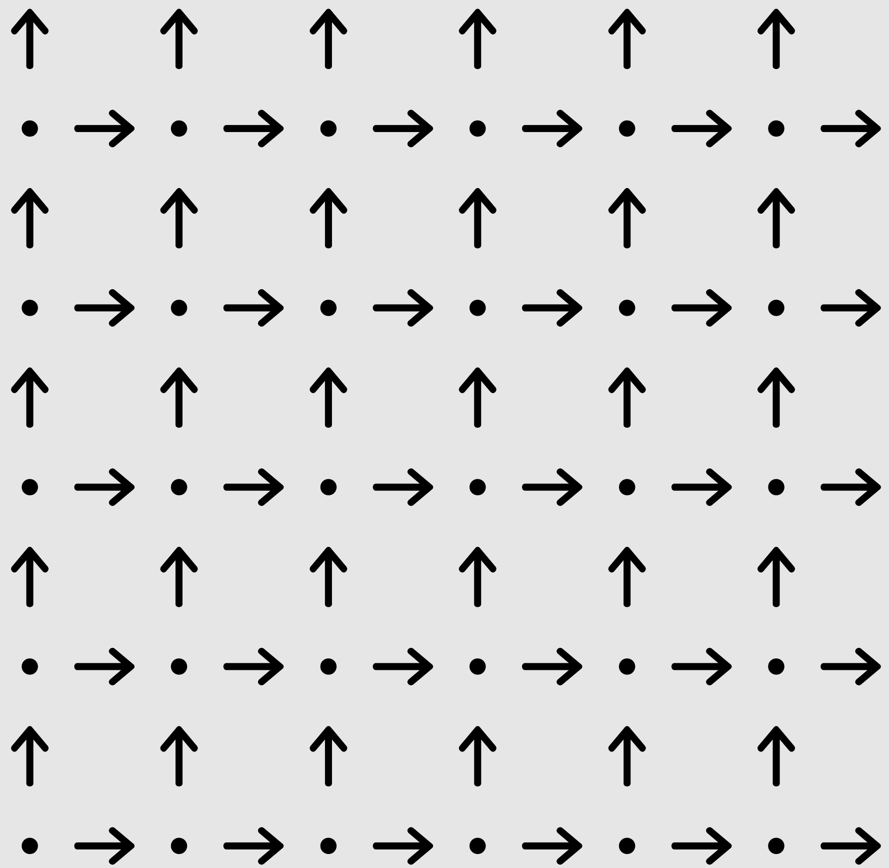
(3) Monotonicity

Multi-parameter persistent homology



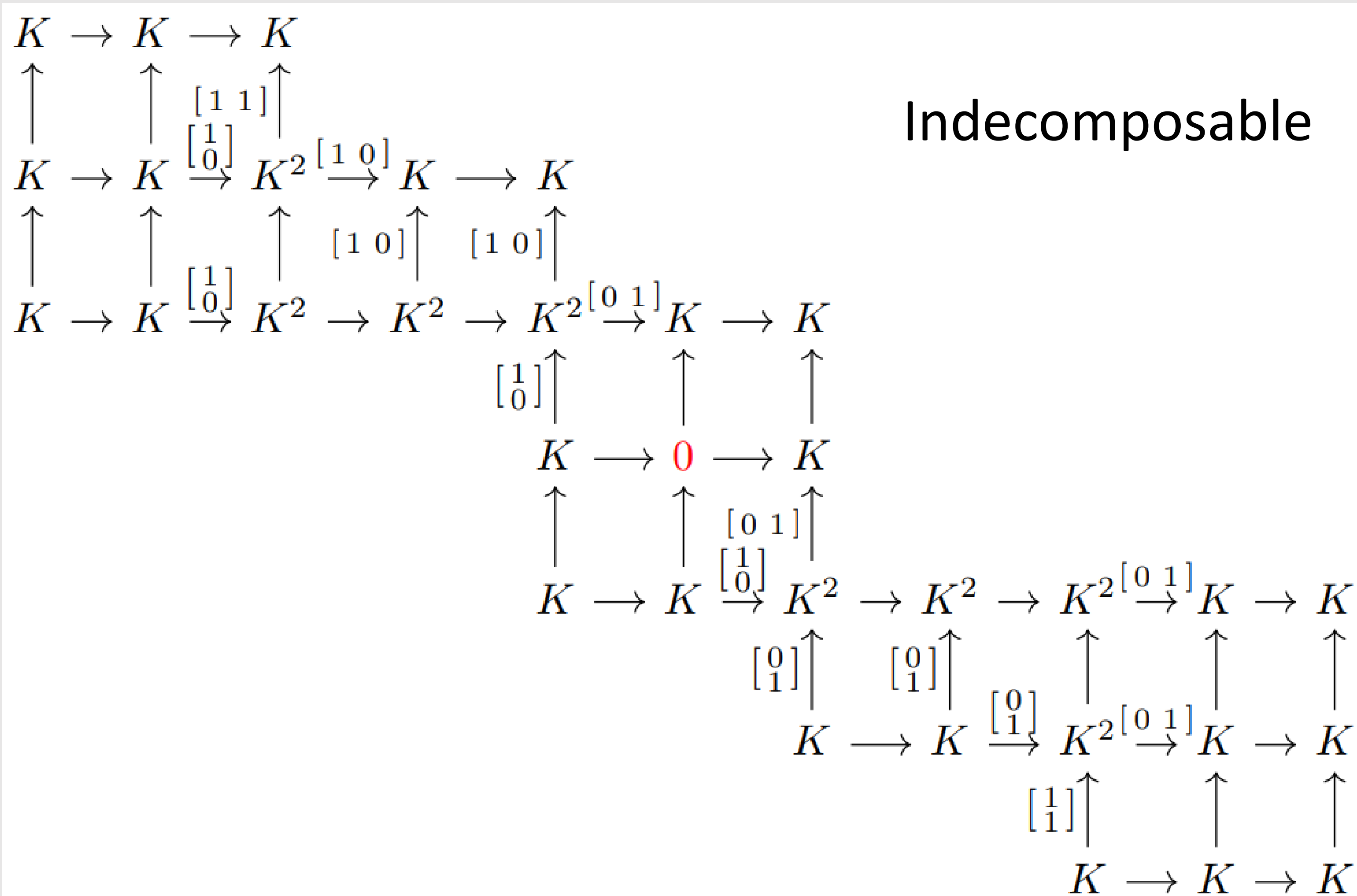
Keller B, Lesnick M, Willke TL. Persistent Homology for Virtual Screening. ChemRxiv. Cambridge: Cambridge Open Engage; 2018

Multi-parameter persistent homology



It is difficult to classify all the indecomposable module
(wild representation type)

Multi-parameter persistent homology



M.Buchet, Emerson G. Escolar "Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module" *Journal of Applied and Computational Topology*

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n \; (a)$

$$\begin{array}{ccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & \circlearrowleft & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$$

$C_{n,m}$

(Modules are always interval decomposable)

We want to understand complex modules (obtained from data) over posets, using “good” modules (interval modules).

Interval approximation

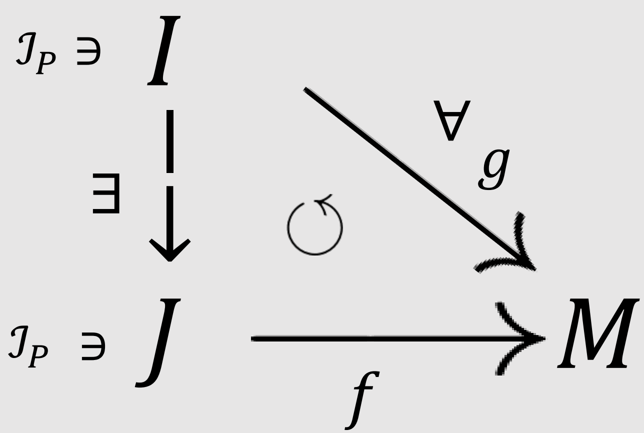
・Benjamin Blanchette, Thomas Brüstle, and Eric J Hanson. “Homological approximations in persistence theory.” *Canadian Journal of Mathematics*, pages 1–38, 2021.

・Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. “Approximation by interval-decomposables and interval resolutions of persistence modules.” *Journal of Pure and Applied Algebra*, 227(10):107397, 2023.

Interval approximation (1/1)

- $\mathcal{J}_P$ is the set of interval decomposable modules over P.
- M is a module over P.

An *interval approximation of M* is a morphism $f\colon J \to M$ with $J \in \mathcal{J}_P$ s. t. for any $g\colon I \to M$ with $I \in \mathcal{J}_P$ factor through f.



An *interval cover* of M is an interval approximation such that the number of direct summands of the domain is smaller than that of other interval approximations (uniquely determined).

Question

How can we calculate interval cover of any modules (easily)?

Question

How can we calculate interval cover of any modules (easily)?

<u>Calculation of interval cover of a module *M*</u>

(1) We have an interval approximation $\bigoplus_{I \in \mathbb{I}(P)} k_I^{m_I} \to M$, where m_I is k-dimension of $\mathrm{Hom}_k(k_I, M)$ and $\mathbb{I}(P)$ is the set of all the intervals in the poset *P*.

(2) We reduce the direct summands of the above interval approximation until we obtain the interval cover.

We give a helpful observation to the Question.

Theorem 2 [Aoki-Escolar-T]

Let P be a finite poset and M be a module over P. For any interval cover of M

$$f = (f_i) : \bigoplus_{i=1}^{n} k_{I_i} \to M,$$

the following holds.

(1) f is surjective.

(2) Each $f_i : k_{I_i} \to M$ is injective.

(3) For every $a \in P$, we have $M(a) = 0$ if and only if $\left(\bigoplus_{i=1}^{n} k_{I_i}\right)(a) = 0$.

In particular, every k_{I_i} is given by an interval submodule of M.

Remark

Recently, [Asashiba, 2023, Proposition 4.8, arXiv:2307.06559] gave the essentially same result (see also [Blanchette-Brüstle-Hanson, Proposition 6.7, 2021, Canadian Journal of Mathematics, 1-38]).

Results

(1)Classification of posets

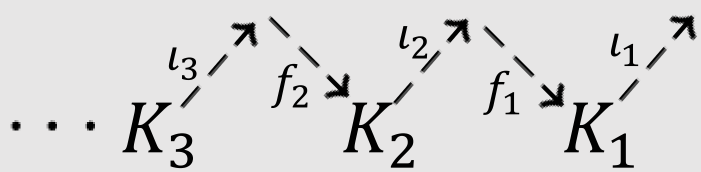
(2)Direct summand injectivity

(3)Monotonicity

Resolution dimension (1/2)

- An *interval resolution of M* is an exact sequence

$$0 \to J_m \xrightarrow{g_m} \cdots \to J_2 \xrightarrow{g_2} J_1 \xrightarrow{g_1} J \xrightarrow{f} M \to 0,$$



then we say that the *interval resolution dimension of M* is m and write int-res-dim $M = m$.

Interval resolution global dimension (2/2)

- *interval resolution global dimension of P* is

$$\text{int-res-gldim}(P) := \sup\{\text{int-res-dim}(M) \mid M\text{: modules over } P\}$$

* [Asashiba-Escolar-Nakashima-Yoshiwaki, 23] show that int-res-gldim(P) < ∞ for any poset finite poset P.

Interval resolution global dimension (2/2)

- *interval resolution global dimension of P* is

$$\text{int-res-gldim}(P) := \sup\{\text{int-res-dim}(M) \mid M\text{: modules over } P\}$$

* [Asashiba-Escolar-Nakashima-Yoshiwaki, 23] show that int-res-gldim(P) < ∞ for any poset finite poset P.

Remark

int-res-gldim(P) is zero if and only if the Hasse diagram of P is either (i) or (ii), where

(i) $\underbrace{1 \longleftrightarrow \cdots \longleftrightarrow n}_{A_n(a)}$, (ii) $\begin{array}{ccccc} & 1 \to 2 \to \cdots \to n & \\ \nearrow & & \searrow \\ \hat{0} & \circlearrowleft & \hat{1} \\ \searrow & & \nearrow \\ & 1' \to 2' \to \cdots \to m' & \end{array}$.

Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P).$$

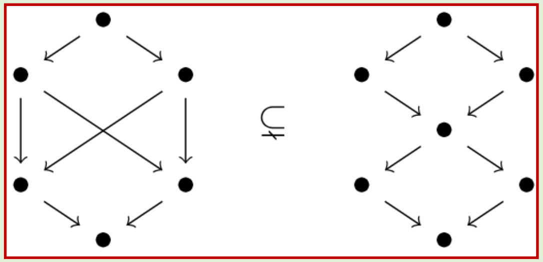
Theorem 3 [Aoki-Escolar-T]

Let P be a finite poset. For any full subposet Q of P, the following inequality holds.

$$\text{int-res-gldim}(Q) \leq \text{int-res-gldim}(P).$$

Remark

The above monotonicity does not hold for (usual) global dimension in general [Igusa-Zacharia, 1990].

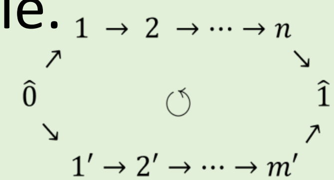


Poset	Q	$\subset$	P	
Global dimension	3	>	2	
Interval global dimension	1	<	2	(over a field with two elements)

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

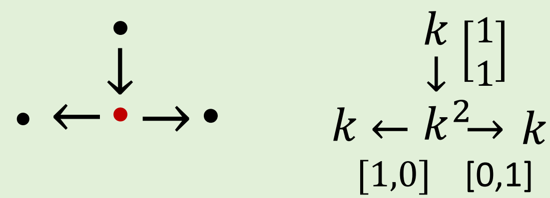
(a) Every module over P is interval decomposable.

(b) The Hasse diagram of P is $1 \longleftrightarrow \cdots \longleftrightarrow n$ or  $C_{n,m}$.

$A_n\,(a)$

Sketch of proof (a ⇒ b)

- The Hasse diagram of P does not have a vertex with degree 3(by Theorem 3).
- P is either A_n or $\tilde{A}_m$ for some n and m.
- P must be A_n or $C_{n,m}$ for some n and m.

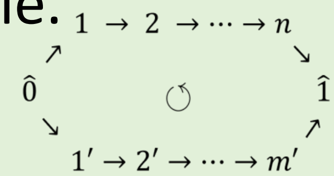


※We prove the converse (interval decomposability of module over $C_{n,m}$) by using theory of special biserial algebra.

Theorem 1 [Aoki-Escolar-T]

Let P be a connected finite poset. The following are equivalent.

(a) Every module over P is interval decomposable.

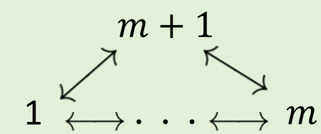
(b) The Hasse diagram of P is $1 \longleftrightarrow \cdots \longleftrightarrow n$ or  $C_{n,m}$.

$A_n\,(a)$

Sketch of proof (a ⇒ b)

- The Hasse diagram of P does not have a vertex with degree 3(by Theorem 3).
- P is either A_n or $\tilde{A}_m$ for some n and m.
- P must be A_n or $C_{n,m}$ for some n and m.

$1 \longleftrightarrow \cdots \longleftrightarrow n$: A_n



: $\tilde{A}_m$

※We prove the converse (interval decomposability of module over $C_{n,m}$) by using theory of special biserial algebra.

Summary

(1) We classified finite posets whose modules are always interval decomposable.

(2) We show that restriction of each direct summand of interval cover is injective.
(It makes calculation of interval cover easier.)

(3) We show the monotonicity of int-res-gldim.
(This is used to show the first result.)

Discussion

- When $C_{n,m}$ is useful in TDA? Stability?
- Does int-res-gldim depend on the characteristic of fields?
- When do we have int-res-gldim(Q) = int-res-gldim(P) for $Q \subset P$?
- Can we calculate interval cover easily?
- Computation using GAP package QPA("Quiver and Path Algebras") and "pmgap" by E. G. Escolar to calculate modules over poset.

Thank you for your attention!

Our paper

reference

- Carlsson, Gunnar, and Vin De Silva. "Zigzag persistence." Foundations of computational mathematics 10 (2010): 367-405.
- Carlsson, Gunnar, Vin De Silva, and Dmitriy Morozov. "Zigzag persistent homology and real-valued functions." Proceedings of the twenty-fifth annual symposium on Computational geometry. 2009.
- Botnan, Magnus, and Michael Lesnick. "Algebraic stability of zigzag persistence modules." Algebraic & geometric topology 18.6 (2018): 3133-3204.
- Asashiba, Hideto. "Relative Koszul coresolutions and relative Betti numbers." arXiv preprint arXiv:2307.06559 (2023).
- Bauer, Ulrich, et al. "Cotorsion torsion triples and the representation theory of filtered hierarchical clustering." Advances in Mathematics 369 (2020): 107171.
- Keller B, Lesnick M, Willke TL. Persistent Homology for Virtual Screening. ChemRxiv. Cambridge: Cambridge Open Engage; 2018.
- Hiraoka, Y., Nakamura, T., Hirata, A., Escolar, E. G., Matsue, K., & Nishiura, Y. “Hierarchical structures of amorphous solids characterized by persistent homology.” Proceedings of the National Academy of Sciences, 113(26), (2016):7035-7040.

reference

- Kiyoshi Igusa and Dan Zacharia. “On the cohomology of incidence algebras of partially ordered sets.” Communications in Algebra, 18(3) (1990):873–887.
- M.Buchet, Emerson G. Escolar“Every 1D Persistence Module is a Restriction of Some Indecomposable 2D Persistence Module” Journal of Applied and Computational Topology.
- Blanchette, Benjamin, Thomas Brüstle, and Eric J. Hanson. "Homological approximations in persistence theory." Canadian Journal of Mathematics (2021): 1-38.
- Magnus Bakke Botnan, Steffen Oppermann, and Steve Oudot. “Signed barcodes for multi-parameter persistence via rank decompositions and rank-exact resolutions.” In International Symposium on Computational Geometry (2021).
- Hideto Asashiba, Emerson G Escolar, Ken Nakashima, and Michio Yoshiwaki. “Approximation by interval-decomposables and interval resolutions of persistence modules.” Journal of Pure and Applied Algebra, 227(10):107397, (2023).
- Assem, Ibrahim, Daniel Simson, and Andrzej Skowronski. “Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory.” Cambridge University Press, (2006).
- 池祐一, E. G. エスカラ, 大林一平，鍛冶静雄「位相的データ解析から構造発見へ」サイエンス社, 2023.

reference

Erdmann, K., Holm, T., Iyama, O., & Schröer, J. “Radical embeddings and representation dimension.” *Advances in mathematics*, *185*(1) (2004):159-177.

Chacholski, W., Guidolin, A., Ren, I., Scolamiero, M., & Tombari, F. (2022). “Effective computation of relative homological invariants for functors over posets.” arXiv preprint arXiv:2209.05923.

Kuhn, Nicholas J. “Generic representations of the finite general linear groups and the Steenrod algebra: II.” *K-theory* 8.4 (1994): 395-428.

McDonald, R Neuhausler, R Robinson, M Larsen, L Harrington, H Bruna, M “Zigzag persistence for coral reef resilience using a stochastic spatial model.” Journal of the Royal Society, Interface volume 20 issue 205 20230280-(23 Aug 2023).