パーシステントホモロジー解析における
区間表現のホモロジー代数的性質

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1 Introduction

We refer the reader to [3] (arXiv:2308.14979) for details on the contents of this article.

Topological data analysis is a rapidly growing field applying the ideas of algebraic topology for data analysis. One of its main tools is persistent homology [1], which can compactly summarize the birth and death parameters of topological features (e.g. connected components, rings, cavities, and so on) of data via the *persistence diagram* or the *barcode*. This allows us to analyze hidden structures in data. Algebraically, one part of the persistent homology analysis can be formalized by using the so-called one-parameter persistence modules, which are just ("pointwise") finite dimensional modules over the incidence algebra of a totally ordered set. In this point of view, one-parameter persistence modules are guaranteed to decompose into the indecomposable modules called *interval modules*, which provide a multiset of intervals that are encoded by the persistence diagram or the barcode.

As a generalization, multi-parameter persistence modules are proposed, understood as representations of n-dimensional grids, and are expected to provide richer information than the one-parameter setting. When dealing with multi-parameter settings, however, there are some difficulties with adapting the same techniques.

Recently, there has been an interest in the use of relative homological algebra in persistence theory. Especially, the notion of interval covers and interval resolutions as a method for dealing with non-interval representations are developed, and the finiteness of the interval resolution global dimension has been confirmed [2].

An aim of this talk is to introduce the properties of interval covers and interval resolutions studied in [3]. First, we give a complete classification of finite posets for which all representations are interval-decomposable. This extends the notion of the above persistence diagrams. Next, we show that the restriction of interval cover of modules to each direct sum is injective. Finally, we show the monotonicity of the interval resolution global dimension. These results suggest a nice behavior of interval representations as invariants in persistence theory.

2 Preliminaries

In this section , we recall the basics of the representation theory of finite dimensional algebras.

2.1 Approximations and resolutions

Let A be a finite dimensional algebra over a field k. We denote by $\mathsf{mod}\,A$ the category of finitely generated right A-modules. Throughout this article, we assume that all modules are finitely generated. For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of A-modules, we denote their composition by $gf\colon X \to Z$. Also, we consider the full subcategory $\mathcal{X} := \mathsf{add}\,\mathfrak{X}$ of $\mathsf{mod}\,A$ for a fixed finite collection $\mathfrak{X}$ of (isomorphism classes of) indecomposable A-modules including all the indecomposable projectives.

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We recall the basic terminology of relative homological algebra.

Definition 1. Let $\mathcal{X} = \mathsf{add}\,\mathcal{X}$ be a full subcategory of $\mathsf{mod}\,A$. For a morphism $f\colon X \to M$ of A-modules, we say that

(1) f is *right minimal* if any morphism $g : X \to X$ satisfying $fg = f$ is an isomorphism.
(2) f is a *right $\mathcal{X}$-approximation* of M if $X \in \mathcal{X}$ and $\mathrm{Hom}_A(Y, f)$ is surjective for any $Y \in \mathcal{X}$.
(3) f is a *right minimal $\mathcal{X}$-approximation* of M if it is a right $\mathcal{X}$-approximation which is right minimal.
(4) A *right minimal $\mathfrak{X}$-resolution* of M is an exact sequence

$$\cdots \longrightarrow J_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{f} M \longrightarrow 0,$$

such that f is a right minimal $(\mathsf{add}\,\mathfrak{X})$-approximation of M, and for each $1 \leq i$, the morphism g_i is a right minimal $(\mathsf{add}\,\mathfrak{X})$-approximation of $\mathrm{Im}\, g_i = \mathrm{Ker}\, g_{i-1}$.
(5) If M has a right minimal $\mathfrak{X}$-resolution of the form

$$0 \longrightarrow J_m \xrightarrow{g_m} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{f} M \longrightarrow 0,$$

then we say that the *$\mathfrak{X}$-resolution dimension* of M is m and write $\mathfrak{X}\text{-res-dim}\, M = m$. Otherwise, we say that the $\mathfrak{X}$-resolution dimension of M is infinity. We set

$$\mathfrak{X}\text{-res-gldim}\, A := \sup\{\mathfrak{X}\text{-res-dim}\, M \mid M \in \mathsf{mod}\, A\}$$

and call *$\mathfrak{X}$-resolution global dimension* of A. Notice that it can be infinity.

2.2 Partially ordered set and its representations

Let P be a finite poset. We recall that the *Hasse diagram* of P is a directed graph whose vertices are in bijection with elements of P and there is a unique arrow $x \to y$ for $x, y \in P$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. The *incidence algebra* $k[P]$ of a poset P is defined to be the quotient of the path algebra of the Hasse diagram of P modulo the two-sided ideal generated by all the commutative relations. The module category $\mathsf{mod}\, k[P]$ can be described in terms of a functor category as follows. Firstly, we regard P as a category whose objects are elements of P, and morphisms are defined by relations in P, i.e., there is a unique morphism $a \to b$ for $a, b \in P$ if and only if $a \leq b$. We denote by $\mathsf{rep}_k(P)$ the category of (covariant) functors from P to the category of finite dimensional vector spaces over k. For V in $\mathsf{rep}_k(P)$, the subset $\mathrm{supp}\, V := \{a \in P \mid V_a \neq 0\}$ is called the *support* of V. The vector $(\dim_k V_a)_{a \in P}$ is called the *dimension vector* of M.

It is well-known that there is an equivalence of abelian categories between $\mathsf{rep}_k(P)$ and the module category $\mathsf{mod}\, k[P]$ of the incidence algebra of P. In this sense, we identify objects V of $\mathsf{rep}_k(P)$ and $k[P]$-modules, and the *support* of a $k[P]$-module M is the subset $\mathrm{supp}(M) = \{a \in P \mid Me_a \neq 0\}$, where e_a is a primitive idempotent of $k[P]$ corresponding to the element $a \in P$.

In our study, the following class of full subposets called interval is basic.

Definition 2. A *full subposet* of P is a subset $P' \subseteq P$ equipped with the induced partial order. Notice that it is completely determined by its elements. We say that

(1) P' is *convex* in P if, for any $x, y \in P'$ and any $z \in P$, $x < z < y$ implies $z \in P'$,
(2) P' is an *interval* of P if P' is connected as a poset and is convex in P.

We denote by $\mathbb{I}(P)$ the set of intervals of P.

The following special class of modules plays an important role in this article.

Definition 3. For an interval I of P, let k_I be a $k[P]$-module given as follows.

$$(k_I)_a = \begin{cases} k & \text{if } a \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad k_I(a \le b) = \begin{cases} 1_k & \text{if } a, b \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

An *interval module* is a $k[P]$-module M such that $M \cong k_I$ for some interval $I \in \mathbb{I}(P)$. Clearly, every interval module is indecomposable.

We denote by $\mathcal{I}_{k,P}$ the set of isomorphism classes of the interval $k[P]$-modules, which is in bijection with $\mathbb{I}(P)$ by $I \mapsto k_I$. Notice that $\mathbb{I}_P$ and $\mathcal{I}_{P,k}$ are finite since so is P. Each module in $\mathsf{add}\,\mathcal{I}_{P,k}$ is said to be *interval-decomposable*. In other words, a given $k[P]$-module M is interval-decomposable if and only if it can be written as

$$M \cong \bigoplus_{I \in \mathbb{I}(P)} k_I^{m(I)}$$

for some non-negative integers $m(I)$. We will write $\mathcal{I}_P$ instead of $\mathcal{I}_{k,P}$ when the base field k is clear.

Since $\mathcal{I}_P$ contains all indecomposable projective $k[P]$-modules by definition, one can consider resolutions by interval modules. By *interval covers over P* (resp., *interval resolutions over P*), we mean right minimal $(\mathsf{add}\,\mathcal{I}_P)$-approximations (resp., $\mathcal{I}_P$-resolutions) of $k[P]$-modules. When the poset P is clear, we may omit it. In addition, we will write

$$\text{int-res-dim}\, M := \mathcal{I}_P\text{-res-dim}\, M \quad \text{and} \quad \text{int-res-gldim}\, k[P] := \mathcal{I}_P\text{-res-gldim}\, k[P],$$

and call them the *interval resolution dimension* of a module M and the *interval resolution global dimension* of $k[P]$ respectively. It has been shown in [2, Proposition 4.5] that the interval resolution global dimension is always finite. To show that, the next proposition is a key.

Proposition 4. [2, Lemma 4.4 and its dual] *The subcategory* $\mathsf{add}\,\mathcal{I}_P$ *is closed under both submodules and quotients of indecomposable modules.*

Then, we can apply [9, Theorem in § 5](cf. [8]) and obtain the following.

Theorem 5. [2, Proposition 4.5] *For any finite poset P,* $\text{int-res-gldim}(k[P]) < \infty$.

3 Results

In this section, we will give three results on rinterval covers and interval resolution dimensions (Theorems 6, 9, and 11).

3.1 Result 1

Firstly, we give a complete classification of finite posets whose modules are always interval-decomposable. This result generalizes the one-parameter settings of persistent homology.

Theorem 6. *Let P be a finite poset and $k[P]$ the incidence algebra of P. Then, the following conditions are equivalent.*

(a) $\text{int-res-gldim}\, k[P] = 0$.

(b) *Every $k[P]$-module is interval-decomposable.*

(c) *Each connected component of the Hasse diagram of P is one of $A_n(a)$ for some orientation a or $C_{m,\ell}$ displayed* *below, where the symbol $\leftrightarrow$ is either $\rightarrow$ or $\leftarrow$ assigned by its orientation a:*

$$A_n(a)\colon \qquad 1 \longleftrightarrow 2 \longleftrightarrow \cdots\cdots \longleftrightarrow n,$$

$$C_{m,\ell}\colon \quad \begin{tikzcd} & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{m-1}"] & m \arrow[dr, "\alpha_m"] & \\ \hat{0} \arrow[ur, "\alpha_0"] \arrow[dr, "\beta_0"'] & & & & \hat{1}. \\ & 1' \arrow[r, "\beta_1"] & \cdots \arrow[r, "\beta_{\ell-1}"] & \ell' \arrow[ur, "\beta_\ell"'] & \end{tikzcd}$$

In particular, these conditions do not depend on the characteristic of the base field k.

We note that equivalences among (a) and (b) in the statement are trivial by definitions.

The following Corollary 7 is immeiate from Theorem 6 by counting intervals in $C_{m,\ell}$.

Corollary 7. *Let A be the incidence algebra of $C_{m,\ell}$. Then, the number of isomorphism classes of indecomposable A-modules is exactly*

$$\frac{m^2 + 4m\ell + \ell^2 + 5m + 5\ell + 6}{2}. \tag{3.1}$$

Example 8. The eleven interval modules over the incidence algebra of the poset $C_{1,1}$ displayed below

$$\begin{tikzcd} & 1 \arrow[dr] & \\ \hat{0} \arrow[ur] \arrow[dr] & & \hat{1} \\ & 1' \arrow[ur] & \end{tikzcd}$$

are

$$\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix},\ \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix},\ \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix},\ \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{smallmatrix},\ \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \end{smallmatrix},\ \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{smallmatrix},$$

where we identify the above dimension vectors with the interval modules. We note that $\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 0 & \end{smallmatrix}$ is not an interval module because the full subposet $\{\hat{0}, 1, \hat{1}\}$ is not convex.

When analyzing data with a filtration of the form of $C_{m,\ell}$, we obtain a persistence module over the poset. By seeing the dimension vector of each indecomposable direct summand of the persistence module, which becomes an interval module by Theorem 6, we can observe the persistence of topological features in the data, similar to the barcodes in standard persistent homology. In this sense, we say that this theorem is an extension of standard persistent homology.

3.2 Results 2

We show the following result.

Theorem 9. *Let P be a finite poset and $\mathcal{I}_P$ the set of isomorphism classes of interval modules. For a given $k[P]$-module M, we take its interval cover $f\colon X = \bigoplus_{i=1}^{m} X_i \to M$, where all the X_i's are interval modules. Then, the following holds.*

(1) *f is surjective.*

(2) *$f|_{X_i}\colon X_i \to M$ is injective for every $i \in \{1, \ldots, m\}$.*

(3) *$\operatorname{supp} X = \operatorname{supp} M$.*

In particular, every X_i can be taken as an interval submodule of M.

An importance of Theorem 9 is that it provides one way to reduce the computational burden for computing interval resolutions. For example, when we compute an interval cover of a module M, the candidates of intervals to be calculated are subsets of $\operatorname{supp} M$. In other words, we do not need to consider intervals that are not included in $\operatorname{supp} M$.

We note that [5, Proposition 4.8] shows Theorem 9 in essentially the same way.

Example 10. We consider the D_4-type quiver $D_4(b)$ displayed below:

$$\begin{tikzcd} & 1 \arrow[d] & \\ 2 & 3 \arrow[l] \arrow[r] & 4. \end{tikzcd}$$

Then, the incidence algebra is just a path algebra of type D_4. The Auslander-Reiten quiver is given by

$$\begin{tikzcd}[column sep=small]
\begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix} \arrow[dr] & & \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \arrow[dr, "a_1"] & & \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \arrow[ur, "b_1"] \arrow[r, "b_2"] \arrow[dr, "b_3"'] & \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \arrow[r, "a_2"] & M \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix}, \\
\begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \end{smallmatrix} \arrow[ur] & & \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \arrow[ur, "a_3"'] & & \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix} \arrow[ur] & &
\end{tikzcd}$$

where all indecomposable modules except for M are interval, but M is

$$\begin{tikzcd} & k \arrow[d, "{{}^t[1\,1]}"] & \\ k & k^2 \arrow[l, "{[1\,0]}"'] \arrow[r, "{[0\,1]}"] & k. \end{tikzcd}$$

Looking at the Auslander-Reiten quiver, we find that an interval resolution of M is

$$0 \longrightarrow \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix} \xrightarrow{{}^t[b_1, b_2, b_3]} \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix} \xrightarrow{[a_1, a_2, a_3]} M \longrightarrow 0,$$

and hence

$$\text{int-res-dim}\, M = 1.$$

Consequently, the interval resolution global dimension for $D_4(b)$ is 1 because M is the only non-interval indecomposable module. By a similar discussion, one can show that any D_4-type quiver has the interval resolution global dimension 1.

3.3 Result 3

Finally, we study a relationship between the interval resolution global dimensions of different posets. Our result is the following.

Theorem 11. *Let P be a finite poset and $k[P]$ the incidence algebra of P. For any full subposet P' of P, we have*

$$\text{int-res-gldim}\, k[P'] \leq \text{int-res-gldim}\, k[P]. \tag{3.2}$$

To put Theorem 11 in motto terms, as the “complexity” of posets P increases the ”complexity” (interval resolution global dimension) of the incidence algebra $k[P]$ increases.

For the usual global dimension, we do not have the above monotonicity in general.

Example 12. Let P and P' be posets given by

$$P : \begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd} \quad \text{and} \quad P' : \begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[d] \arrow[drr] & & \bullet \arrow[d] \arrow[dll] \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

respectively. Then, P' is a full subposet of P, which is obtained by removing the point in the center. The global dimension of $k[P]$ is 2 and that of $k[P']$ is 3 (over an arbitrary field), see [7, Section 3]. On the other hand, we have $\text{int-res-gldim}\, k[P'] = 2 \leq 3 = \text{int-res-gldim}\, k[P]$ over a field with two elements.

■Acknowledgement This work is supported by JSPS Grant-in-Aid for Transformative Research Areas (A) (22H05105). S.T. is supported by JST SPRING, Grant Number JPMJSP2148.

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