On Interval Global Dimension of Posets: a Characterization of Case 0

青木 利隆 (神戸大学大学院人間発達環境学研究科)[*1]
ESCOLAR, Emerson Gaw (神戸大学大学院人間発達環境学研究科)[*2]
多田 駿介 (神戸大学大学院人間発達環境学研究科)[*3]

June 15, 2023

Abstract

We study the relative homological algebra of posets with respect to the intervals. We introduce our recent research on the properties of the supports of interval approximations and interval resolution global dimension. We also provide necessary and sufficient conditions on a poset to ensure that any representation is interval decomposable (i.e. a characterization of the case where interval resolution global dimension is equal to 0).

1 Introduction

Topological data analysis is a growing field in data science and has been applied in various areas to analyze complex and high-dimensional data. *Persistent homology (PH) analysis* is one of the main techniques in topological data analysis [1].

In PH analysis, a sequence of topological spaces or simplicial complexes equipped with inclusion maps, called a filtration, is constructed from a given point cloud data. PH analysis then describes the changes in the topological properties like the number of connected components, holes, cavities, and so on, in the filtration. The topological features that persist in the filtration are considered to be interesting, and they can provide new insights into data to uncover hidden information in the data.

Algebraically, the above can be explained in the following way. By taking the singular homology with coefficients in a field, a sequence of vector spaces and linear maps are induced. This sequence of vector spaces and linear maps is called a one-parameter persistence module and can be understood as a representation of a finite totally ordered set. Gabriel's theorem states that these modules can be decomposed into *interval modules*, and each interval module is parametrized by two non-negative numbers $b \leq d$. In this context, interval modules describe the birth b, death d, and the persistence $d - b$ of a topological feature.

Persistence modules are understood as functors from a poset $(P, \leq)$ to the category of finite dimensional vector spaces over a given field k. Over a finite poset P, a module is said to be an *interval module* if it is isomorphic to a module M whose support is a connected and convex full subposet of P, and where M takes on the one-dimensional

[*1] e-mail: toshitaka.aoki@people.kobe-u.ac.jp
[*2] e-mail: e.g.escolar@people.kobe-u.ac.jp
web: https://emerson-escolar.github.io/
[*3] 〒 657-8501 兵庫県神戸市灘区鶴甲 3-11 神戸大学大学院人間発達環境学研究科
e-mail: kyorochan1357@gmail.com

This work is supported by JSPS Grant-in-Aid for Transformative Research Areas (A) (22H05105). The third author is supported by JST SPRING, Grant Number JPMJSP2148.
2020 Mathematics Subject Classification: 55N31, 18G20 (primary); 18G25, 16G20 (secondary).
Keywords: persistence modules, incidence algebras, interval modules, relative homological algebra.

vector space k and identity maps over its support and zero elsewhere. A module is said to be *interval decomposable* if it is isomorphic to a finite direct sum of interval modules. Note that in general, persistence modules are not always interval decomposable.

Thus, there are attempts to capture persistent topological features in this general setting. Our study is inspired by the work [2] introducing homological algebra in persistence theory, and the work [3]. Following these works, we assume the poset P is finite, and identify persistence modules as modules over the incidence algebra $k[P]$.

In this work, we first show the following.

Theorem 1.1. *Let A be a finite dimensional k-algebra and $\mathscr{I}$ be a subcategory of $\mathsf{mod}\, A$ which is closed under taking finite direct sums, summands and isomorphisms, and satisfies the property that any quotient of an indecomposable object of $\mathscr{I}$ is also in $\mathscr{I}$. For an A-module M, if it admits a right minimal $\mathscr{I}$-approximation $f\colon X \to M$, then the following hold.*

1. *The restriction of f to each indecomposable direct summand of X is injective.*

2. $\operatorname{supp} X \subseteq \operatorname{supp} M$.

For example, the subcategory of interval decomposable modules $\mathscr{I}_P$ over a finite poset P satisfies the hypothesis of the Theorem with $A = k[P]$ and $\mathscr{I} = \mathscr{I}_P$. As an application, we obtain the following.

Theorem 1.2. *If Q is a convex subposet of P, then the interval resolution global dimension of the incidence algebra $k[Q]$ is less than or equal to that of $k[P]$, i.e., we have the following inequality*

$$\mathscr{I}_Q\text{-res-gldim } k[Q] \leq \mathscr{I}_P\text{-res-gldim } k[P]. \tag{1}$$

Finally, we prove the following.

Theorem 1.3. *Let P be a finite poset. The following are equivalent.*

1. $\mathscr{I}_P$-res-gldim $k[P] = 0$ *(i.e. any interval module is interval decomposable).*

2. *The Hasse diagram of P is a Dynkin quiver of type A or a quiver of the form*

$$\begin{tikzcd}
& 1 \arrow[r] & \cdots \arrow[r] & m \arrow[dr] & \\
\bar{0} \arrow[ur] \arrow[dr] & & & & \bar{1} \\
& 1' \arrow[r] & \cdots \arrow[r] & \ell' \arrow[ur] &
\end{tikzcd}$$

for some positive integers m and ℓ.

References

[1] Edelsbrunner, H., Letscher, D., & Zomorodian, A. (2002). Topological persistence and simplification. Discrete & computational geometry, 28, 511-533.

[2] Blanchette, B., Brüstle, T., & Hanson, E. J. (2021). Homological approximations in persistence theory. Canadian Journal of Mathematics, 1-38.

[3] Asashiba, H., Escolar, E. G., Nakashima, K., & Yoshiwaki, M. (2023). Approximation by interval-decomposables and interval resolutions of persistence modules. Journal of Pure and Applied Algebra, 227(10), 107397.