P-23 Interval resolution global dimension and interval cover

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This poster presentation is based on a joint work with Toshitaka Aoki, Emerson G. Escolar.

1 Introduction | Persistence modules

We study the relative homological algebra of posets with respect to the intervals. We introduce our recent research on the

(1) supports of interval approximations and

(2) properties of interval resolution global dimension.

(3) necessary and sufficient conditions on a poset to ensure that any representation is interval decomposable.

(Third statement is equivalent to a characterization of the case where interval resolution global dimension is equal to 0)

Notation

Let P be a finite poset, A:=$k[P]$ be the incidence algebra over P.

2 Definitions

・Interval modules

For an interval I of P, let k_I be a $k[P]$-module given as follows.

$$(k_I)_a = \begin{cases} k & \text{if } a \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad k_I(a \le b) = \begin{cases} 1_k & \text{if } a, b \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{X}$ be set of isomorphism classes of interval modules of A=$k[P]$.

・Interval resolution global dimension

For a morphism $f\colon X \to M$ of A-modules, we say that

1. f is *right minimal* if any morphism $g : X \to X$ satisfying $fg = f$ is an isomorphism.
2. f is a *right $\mathcal{X}$-approximation* of M if $X \in \mathcal{X}$ and $_A(Y, f)$ is surjective for any $Y \in \mathcal{X}$.
3. f is a *right minimal $\mathcal{X}$-approximation* of M if it is a right $\mathcal{X}$-approximation which is right minimal.

If we can take a sequence of right minimal $\mathcal{X}$ -approximation, the infimum of the length of the sequence, denote int-res-dim(M), is called the *interval resolution dimension of M.*

$$\begin{tikzcd} \cdots \arrow[r] & J_2 \arrow[rr, "g_2"] \arrow[dr, "f_2"'] & & J_1 \arrow[rr, "g_1"] \arrow[dr, "f_1"'] & & J \arrow[r, "f"] & M_A \arrow[r] & 0 \\ & & K_2 \arrow[ur, "\iota_1"'] \arrow[u, phantom, "\circlearrowleft"] & & K_1 \arrow[ur, "\iota"'] \arrow[u, phantom, "\circlearrowleft"] & & & \end{tikzcd}$$

3 main theorems

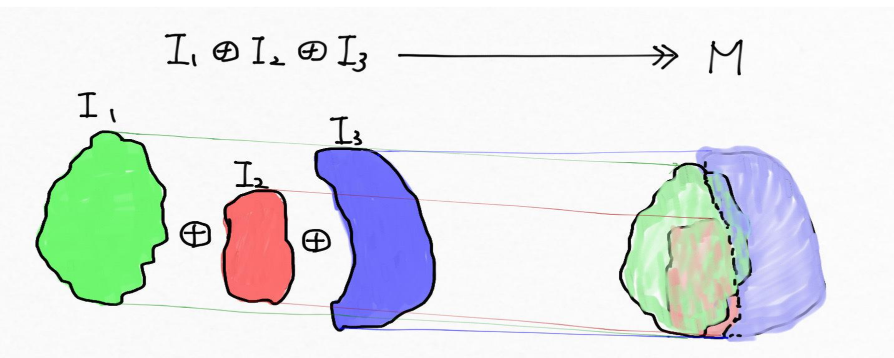
(i) Summand-injectivity / Support of interval cover

Let

$$(f_i)_{i=1}^m\colon \bigoplus_{i=1}^m k_{I_i} \longrightarrow M$$

be a right minimal $\mathcal{X}$-approximation of M, where $X_i \in \mathcal{X}$ are indecomposable. Then, we have following.

1. Each $f_i : k_{I_i} \to M$ is injective.
2. $\mathrm{supp}(X) = \mathrm{supp}(M)$.



(ii) Monotonicity

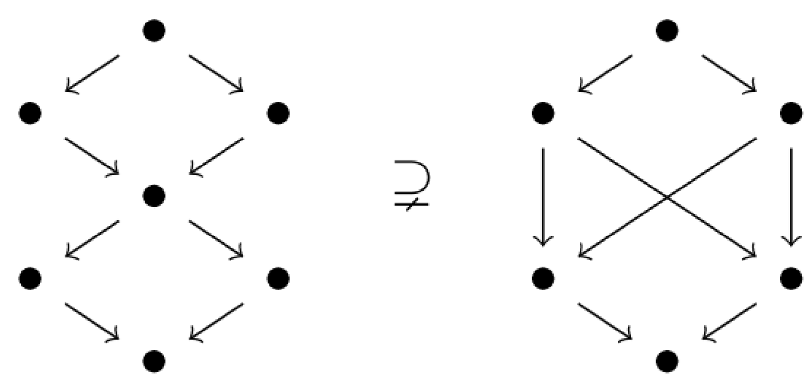
Let P be a finite poset. For any full subposet P' of P, the following inequality holds .

Int-res-gldim $k[P'] \leqq$ int-res-gldim $k[P]$.

Remark

Let P' be a subposet of P like on the right.

The global dimension of $k[P]$is 2 and that of $k[P']$ is 3 (over an arbitrary field), see [4, Section 3]



P P'

(iii) When are all modules interval decomposable?

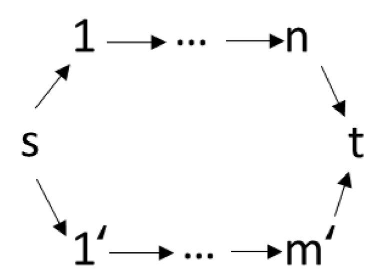
Let P be a poset with n vertices and $k[P]$ the incidence algebra of poset over a field k. Then, the following conditions are equivalent.

1. Every $k[P]$-module is interval-decomposable.
2. Every indecomposable $k[P]$ is interval.
3. int-res-gldim $k[P] = 0$.
4. P is either $A_n(a)$ for some orientation a or $C_{m,\ell}$ for some positive integers $m, \ell > 0$ with $m + \ell = n - 2$.

All indecomposable modules are interval over the posets

$$1 \longleftrightarrow \cdots \longleftrightarrow n$$

$A_n(a)$



$C_{m,\ell}$

4 Discussion

・Does interval resolution global dimension depend on characteristic of fields?
・Can we apply our result (iii) and the posets $C_{m,\ell}$ to persistent homology analysis?

5 Reference

[1] Toshitaka Aoki , Emerson G. Escolar and Shunsuke Tada. Summand-injectivity of interval approximations and monotonicity of interval global dimension. Preprint

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[3] Benjamin Blanchette, Thomas Brüstle, and Eric J Hanson. Homological approximations in persistence theory. Canadian Journal of Mathematics, pages 1–24, 2021.

[4] Kiyoshi Igusa and Dan Zacharia. On the cohomology of incidence algebras of partially ordered sets. Communications in Algebra, 18(3):873–887, 1990.