Tensor Prime Ideals and Persistence Modules

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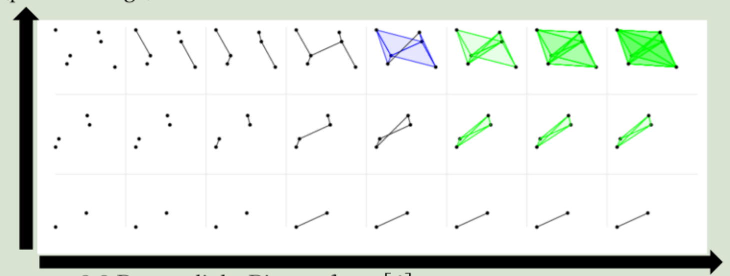
(1) Can we give new invariants for multi-parameter persistent modules?

Multi-parameter persistence modules are derived from a multi-filtration of data (point cloud) as well as one-parameter persistence modules. For example, two-parameter persistence modules may arise from bifiltrations of data to account for the partial charge of atoms. In fact, B. Keller et al. have studied drug discovery suggesting a possibility to extract information that would not be available with one-parameter persistence modules [4].



2,3 Butanediol Picture from [4]

It is well known that, unlike one-parameter persistence modules, multi-parameter persistence modules are complex. **Obtaining information about data from multi-parameter persistence modules is a challenge in this field.**

Data ➡ Multi-filtration
➡ Persistence module $\xrightarrow{?}$ Persistence diagram

To face this challenge, we studied what is called tensor prime ideals **(3)** to obtain useful invariants from persistence modules under a simple condition, see **(2)**. It is a tool in a related field of algebraic geometry called tensor triangulated geometry (TTG).
Often the theory of algebraic geometry is applied to multi-parameter persistence modules. We hope TTG also relates to the theory of persistence module.

(2) Persistence modules

We deal with persistence modules, functors from certain directed graph (also called quivers of type A) to the category of finite-dimensional vector spaces. The directed graph is assumed to have bounded path lengths for the sake of simplicity.

$$\cdots \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \cdots$$

$$\xcancel{\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots}$$

(3) Prime tensor ideals

Persistence modules can be added (direct sum) and multiplied (tensor product). Its tensor product is defined by

$$\begin{gathered}\cdots W_{i-1} \leftrightarrow W_i \leftrightarrow W_{i+1} \leftrightarrow \cdots \\ \bigotimes \\ \cdots \leftrightarrow V_{i-1} \leftrightarrow V_i \leftrightarrow V_{i+1} \leftrightarrow \cdots \\ \| \\ \cdots \leftrightarrow V_{i-1} \otimes W_{i-1} \leftrightarrow V_i \otimes W_i \leftrightarrow V_{i+1} \otimes W_{i+1} \leftrightarrow \cdots\end{gathered}$$

Prime tensor ideals are similar to that of prime ideals in ring theory. In the category of persistence modules ($\mathcal{A}$), tensor products, direct sums and tensor prime ideals($\mathcal{P}$) are determined.

	Ring	Category of persistence modules ($\mathcal{A}$)
Sum	$+$	$\bigoplus$
Product	$\cdot$	$\bigotimes$
Prime ideal	Prime ideal	Tensor prime ideal ($\mathcal{P}$)

Tensor prime ideals have the following properties similar to those of prime ideals of a ring.

$$V, W \in \mathcal{P} \Longrightarrow V \oplus W \in \mathcal{P}$$
$$V \in \mathcal{A}, W \in \mathcal{P} \Longrightarrow V \otimes W \in \mathcal{P}$$

There are other technical conditions of tensor prime ideals omitted here.

(4) Results

The prime tensor ideals in the category of persistence modules are in canonical bijection with prime ideals of the ring $\prod_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})_i$

$$\begin{aligned}\mathrm{Spc}(\mathcal{A}) &\longrightarrow \mathrm{Spc}(\textstyle\prod_{i\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})_i) \\ \mathcal{P} &\longmapsto \{(a_i)_{i\in\mathbb{Z}} : a_i = 0 \text{ iff } V_i = 0 \text{ where } V \in \mathcal{P}\}\end{aligned}$$

The space of these tensor prime ideals and the space of prime ideals with Zariski topology are homeomorphic by the above bijective map.

(5) Remark

If the length of paths in the directed graph is not bounded, like

$$\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$$

then, the above map in **(5) Results** is not surjective.

(6) Discussion

It would be nice if there is a relation between tensor triangulated geometry (TTG) and multi-parameter persistence modules (or topological data analysis). So, our questions are as follows,

- Can we study persistence modules by TTG?
- Can we study TTG through the theory of persistence modules?

Acknowledgements

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References

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